

New class of connected topological spaces

By

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المستخلص :-

تم في هذا البحث دراسة خصائص لصنف أكثر توسعا" من الفضاء المترابط وهو الفضاء (perfect connected) وكذلك تم دراسة التطبيق (magic-almost open) وتمت البرهنة على إن (perfect connected)

كل تطبيق حقيقي معرف على perfect connected يكون تطبيق Magic-almost open فضاء

وكذلك تم تعريف التطبيق المفتوح تقريبا" والاستفادة منه في البرهنة على إن كل تطبيق حقيقي معرف على فضاء (perfect connected) يكون مستمر تطبيق تشاكل هوميومورفزم .

Abstract :-

In this paper we study the properties a large class of connected topological spaces : the class is perfect connected spaces. Also we study the magic-almost open map and we prove every real continuous map defined on a perfect connected spaces then its magic- almost open map. Also we defined the almost- open map [1] and we used to prove for every real continuous map defined on a perfect connected space then the map is homomorphism.

Definition (1):-A topological space (X,T) is said to be perfect connected space if there exist a connected and locally connected space Y and a continuous onto map $f: X \rightarrow Y$.

Remark (1):-Let Z be non empty subset of the topological space X we say that the set Z is perfect connected space if Z with relative topology is perfect connected set.

Example (1) :-The usual topological space (\mathbb{R}, U) is perfect connected space since the topological space (\mathbb{R}, U) is connected and locally connected space and we can defined the identity map $I : \mathbb{R} \rightarrow \mathbb{R}$ which is onto continuous map.

Definition (2) :-Let X and Y be two- topological space the onto map $f : X \rightarrow Y$ is said to be almost open map iff for all point $y \in Y$. There exists a point $x \in f^{-1}(y)$ such that f is interior map in a point x . [3]

Lemma (1) :-The map $f : X \rightarrow Y$ is almost open map iff for all $y \in Y$ then $f^{-1}(y) \cap \text{int } f \neq \emptyset$.

Example : (2) :- Let $X = \{a, b, c\}$ and $T_x = \{\emptyset, x, \{a\}, \{c\}, \{a, c\}\}$ be the topological defined on X and let $Y = \{x, y\}$ and $T_y = \{\emptyset, y, \{x\}\}$ be the topological defined on y .

Let $f: X \rightarrow Y$ be the map definition by : $f(b) = f(c) = y \ \exists \ f(a) = x$

Since $f^{-1}(y) \cap \text{int } f \neq \emptyset \ \forall \ y \in Y$ then by the lemma (1-1-8) f is almost open map.

Remark: (2):- If $\{X_i\}$ is a family of subsets of perfect connected space on the topological space X and $\bigcap_{i \in I} X_i \neq \emptyset$ then $\bigcup_{i \in I} X_i$ is completely perfect set.

Remark (3):-

- 1) Every perfect connected space is connected space.
- 2) Every connected space and locally connected is completely perfect space.

Definition (3):- Let $f: X \rightarrow R$ be a map defined on the topological space X then f is called magic-almost open map if every point $t \in f(x)$ there exist a non empty subset $K_t \subseteq f^{-1}(t)$ (Card $K_t \leq 2$) such that every neighborhood $K_t \subseteq U$ then $f(a)$ is neber hood the point t .

Remark (4):- The subset A of the topological space X is open and closed set iff

$$b(A) = \emptyset$$

Theorem (1):- Let (X, T) be a topological space and X^* be a connected subset on X and let the points x_0, x_1 belong to X^* the map $g: X \rightarrow R$ definition on X then:

I- x_0 is local max point (local min point) on the map g .

II- $g(x_0) < g(x_1)$.

III- g is continuous map for every point on X^* then there exist a point

$x^* \in X^*$ Satisfy the following:

1- $g(x^*) = g(x_0)$.

2- x^* Is not local max point (local min) for a map g

3- x^* is not local min point (local max) for a map g if for all open set $\Omega \subseteq X$, $\Omega \cap X^*$ there exist the point $\bar{x} \in \Omega$ such as that $g(\bar{x}) \neq g(x_0)$

Proof: - Since the map has a local max in the point x_0 , \exists the neighborhood V for point x_0 such that $g(x) \leq g(x_0) \forall x \in V$. Let F be the set family of neighborhood which satisfy $g(x) \leq g(x_0)$, since $V \in F$ then $F \neq \emptyset$. let $V^* = \bigcup V$ so that $V^* \in F$ and since $x_0 \in V^*$, $x_0 \in X^*$ then $x_0 \in V^* \cap X^*$ and $V^* \cap X^* \neq \emptyset$. $x_1 \in X^*$ and $x_1 \in V^*$ so that $X^* - V^* \neq \emptyset$ and X^* is subset connected space then X^* is the only open and closed set, hence $b(V^* \cap X^*) \neq \emptyset$, Now if $b(V^* \cap X^*) = \emptyset$ then $V^* \cap X^*$ is open and closed set (2-1-3) and there exist subset $V^* \cap X^*$ from X^* is open and closed set which is contradiction.

Let $x^* \in b(V^* \cap X^*)$ and let U be neighborhood for the point x^* hence $U \cap (V^* \cap X^*) \neq \emptyset$ and $U \cap (X^* - V^*) \neq \emptyset$.

Proof (1): Let $K = \{x \in X^* : g(x) \leq g(x_0)\}$ because $g(x) \leq g(x_0)$ is satisfy for all neighborhood contained the point x^* and g is continuous map then $g(x^*) \leq g(x_0)$. assume $g(x^*) < g(x_0)$ and Let W be a neighborhood contained the point x^* and $W^* = V^* \cap W$ Since $V^* = \bigcup V$ and F be a family of all open neighborhood which satisfy $g(x) \leq g(x_0)$ So that $W^* \in F$ and there for $W^* \subseteq V^*$. For all neighborhood N to x^* contains all points from $V^* \cap X^*$ and the point $\bar{x} \in X^* - V^*$, W is contain at least a point y from $V^* \cap X^*$ and a point \bar{y} from

$X^* - V^*$, So that $\bar{y} \in (X^* - V^*) \cap W$ and $\bar{y} \in (X^* - V^*)$, $\bar{y} \in W$. since

$W \subseteq V^*$ then $y \notin W$ that is contradiction. Hence $g(x^*) = g(x_0)$.

Proof (2): Assume x^* is a local min point for a map g then \exists a neighborhood contains a point x^* such that $g(x) \geq g(x^*) \forall x \in S$ since $g(x^*) = g(x_0)$ then $g(x) \leq g(x_0) \forall x \in S$ and $S \subseteq V^*$ because

$\forall N$ for x^* contains the points for $V^* \cap X^*$ and point y_1 , from $X^* - V^*$ we get $y_1 \in (x^* - V^*) \cap S$ and $y_1 \in (X^* - V^*)$ and $y_1 \in S$ so that $y_1 \notin S$ is contradiction .Hence X^* is not local max point for map g .

Proof (3): Assume X^* is local minim point for a map g then a neighborhood S' contains point x^* such that $g(x) \geq g(x^*)$. $\forall x \in S'$ Let $\Omega = V^* \cap S'$ then $\Omega \cap X^* \neq \emptyset$ by assume $\exists \bar{x} \in \Omega \ni g(\bar{x}) \neq g(x_0)$ because $x \in S'$ then $g(\bar{x}) > g(x_0)$ and $\bar{x} \in V^*$ then $g(\bar{x}) \leq g(x_0)$ that is contradiction so that x^* is not local min point for a map g by the same method if x is local min point for a map g and $g(x_0) > g(x_1)$.

Theorem (2):- Every real continuous map defined on perfect connected space then it's magic – almost open map.

Proof: - Let X is perfect connected space and $f: X \rightarrow \mathbb{R}$ be continuous real map and $T = f(x)$ assume $T = \{t\}$ now to prove f is magic – almost open map. Since

$f(x) = \{t\}$ then $f^{-1}(t) = X$ and \exists a non empty subset $K_t \subseteq X \ni \text{card } K_t \leq 2$ So that \exists a non empty subset $K_t \subseteq f^{-1}(t) \ni \text{card } K_t \leq 2$, then for all neighborhood $K_t \subseteq U$, $f(U)$ is a neighborhood for a point t . now assume that the interval T is not vanishing since X is perfect connected space then \exists the space Y connected and locally connected and the map $h: Y \rightarrow Y$ is continuous map.

Let $g = f \circ h$ and $t \in T$. Assume $\inf f(x) < t < \sup f(x)$, by theorem (2-1-4) $y^* \in Y$ sub such that $g(y^*) = t$ and y^* is not locally max point for a map g and $\exists y^* \in Y$ such that $g(y^*) = y$ and y^* is locally min point for a map g let U^* be a neighborhood contains the point y^* since Y is Perfect connected

Then \exists a connected neighborhood $V^* \ni y^* \in V^* \subseteq U^*$ and $t \in g(V^*) \subseteq g(U^*)$, $[t, t + \epsilon] \subseteq g(U^*) \quad \forall \epsilon > 0$ Because Y is perfect connected space then for all neighborhood U^{**} is contained the point y^* and \exists connected a neighborhood $V^{**} \ni y^{**} \in V^{**} \subseteq U^{**}$. then $t \in g(V^{**}) \subseteq g(U^{**})$,

$[t - \epsilon, t] \subseteq g(U^{**}) \quad \forall \epsilon > 0$ and \forall neighborhood N contained the set $\{$

y^*, y^{**} }. Then $g(N)$ is neighborhood contained the point t . let $K_t = \{h(y^*), h(y^{**})\}$ and W be a neighborhood contained the set K_t since h is continuous map then \exists an open neighborhood W_1 contained the point y^* and $h(W_1) \subseteq W$ also \exists open neighborhood W_2 contained the point Y^{**} and $h(W_2) \subseteq W$ let $U = W_1 \cup W_2$ then $h(U) = h(W_1) \cup h(W_2)$ and $h(U) \subseteq W$ therefore $f(h(U)) \subseteq f(W)$ i.e. $g(U) \subseteq f(W)$ and hence f is magic – almost open map .

now assume $t = \inf f(X)$ by theorem (2-1-4) $\exists y^* \in Y \ni g(y^*) = t$ and y^* is not locally max point for map g let $K_t = \{h(y^*)\}$ and W is a neighborhood contained the set K_t because h is continuous map then \exists an open neighborhood W_1 contained the point y^* and $f(h(W_1)) \subseteq f(W)$ i.e. $g(W_1) \subseteq f(W)$ there fore f is magic –almost open map by same method we prove $t = \sup f(X)$.

Remark (4):-When card $K_t = 1$ the magic –Almost open map is almost open map.

Theorem (3):-Let X is a perfect connected space and $f: X \rightarrow R$ be a one to one continuous map then f is homomorphism.

Proof: we must to prove f^{-1} is continuous map i.e. to prove f is open map. because f is continuous map and X is perfect connected by theorem (2-1-5) f is magic–Almost open map, and since f is one to one map then $\forall t \in f(X)$, $f^{-1}(t)$ is only one element that is mean card $K_t = 1 \forall t \in f(X)$ by Remark (2-1-2) f is almost open map $f^{-1}(t) \cap \text{int } f \neq \emptyset \forall t \in f(X)$ and $f^{-1}(t) \subseteq \text{int } f$ also \forall open set U in $X \ni f^{-1}(t) \in U$, t is interior point in $f(U)$ i.e. \exists open set $G \ni t \in G \subseteq f(U) \forall t \in f(X)$ is interior map in $f^{-1}(t)$ then f is interior map for every point of X i.e. in $f = X$ hence f is open map there for f is homomorphism .

Remark (5):- In the next example we prove if $f: X \rightarrow \mathbb{R}$ is magic –Almost open map then f it's not necessary X is perfect connected.

Example (3):- let X be countable connected hausdorff space open set A which homeomorphism with the rational number space. It is easy to show that every continuous map $f: X \rightarrow \mathbb{R}$ is constant, so that f is magic –Almost open map and to show that let $t \in f(x)$, since f is constant map then $f^{-1}(t) = X$. then there exist a non empty subset $K_i \subseteq X \ni \text{card } K_i \leq 2$ and a non empty subset $K_i \subseteq f^{-1}(t) \ni \text{card } K_i \leq 2$ such that for each neighborhood $K_i \subseteq U$ then $f(u)$ is neighborhood of the point t . and hence f is magic –Almost open map.

Now, if we suppose X is perfect connected space ,then by definition (2-1-1) there exist a connected and locally connected space Y with the continuous on to map $f: X \rightarrow Y$, implies that $g(Y) = X$. let z be the point in A and $B = g^{-1}(Z)$. Because X is hausdorff space then $\{z\}$ is closed space ,and since g is continuous map then $B = g^{-1}(Z)$ is closed set in Y . let $b \in B$ because A is open set and g is continuous map then \exists an open set U contained the point b such that $g(U) \subseteq A$, since V is connected space and g is continuous map then $V \subseteq g^{-1}(z)$ and $b \in V \subseteq B = g^{-1}(z)$. Hence B is connected that is contradiction.

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