HOC FINITE DIFFERENCE SCHEME TO FIND THE NUMERICAL SOLUTION OF STEFAN PROBLEM

Proof. Dr. R. S. Naoum and S.Q. Abbas***

الخلاصة

في هذا البحث ناقشنا الحل العددي لمسائلة "ستيفن" و قيمنا طرق الحل من خلال تطبيقها على أنواع مختلفة من هذه المسائل، مع وصف لطبيعة المسئلة قيد البحث ، و كذلك الحل التحليلي لها . وقمنا بمناقشة الحل و استقراره مستخدمين أسلوب(HOC) وهو نظام الفروقات المنتهية ذو الرتبة العالية المتراص مبينين طبيعة أسلوب الحل من خلال التطبيق المباشر على المسئلة قيد البحث.

ABSTRACT

In this paper, the numerical solution of Stefan problem was discussed , assess their effectiveness over the range of Stefan type problems and containing a general statement of the Stefan problem, the description of a test problem and its analytic solution are also given. The Hoc finite difference formulation to find the numerical solution is outlined, with it stability, and the problems associated with its direct implementation are highlighted via a test problem. .

Keyword: Stefan problem, Height order compact (*HOC*), Parabolic P.D.E

1-Introduction

The Stefan problem, free boundary problem, in one space dimension can be described as follow: In a region $\Omega = \{(y,t) | t > 0, 0 < y < s(t)\}$, a function u is sought as the solution of the linear heat equation

$$u_t - u_{yy} = f \ \ in \ \Omega .$$

The initial temperature $u(y,0)$ and $u_y(0,t)$ are prescribed. The free boundary $y = s(t)$ is defined by the condition $u(s(t),t) = 0$, on the one hand and the additional condition $s_t + u_y(s(t),t) = 0$, on the other. The melting (or freezing) of a block of ice, reducing the water in the dam, is a two examples of physical interpretation.

The numerical formulation of Stefan problem which we considered is as follow:

*Department of Mathematics, College of Science, University of Baghdad. E-male Reyadh_Naoum@Yahoo.com.
**Department of Mathematics, College of Science, University of Baghdad. E-male saadqasim75@Yahoo.com.

Problem ***P^v***: Given $T_0 > 0$, and *g(y)* for $y \in I = (0,1)$ with $g'(0) = 0$, $g(1) = 0$. Find $\{s(\tau), v(y,\tau)\}$ such that

$$s(t) > 0 \text{ for } 0 < t < T_0,\ s(0) = 1, \tag{1.1}$$

$$v_{yy} - v_\tau = 0 \text{ in } \Omega = \{(y,\tau) | 0 < \tau < T_0, 0 < y < s(\tau)\}, \tag{1.2}$$

$$\left.\begin{array}{l} v_y(0,\tau) = 0 \\ v(s(t),\tau) = 0 \end{array}\right\} \text{for } 0 < \tau \le T_0, \tag{1.3}$$

$$v(y,0) = g(y) \text{ for } y \in I, \tag{1.4}$$

and in addition

$$\frac{ds}{d\tau} + v_y(s(\tau),\tau) = 0 \text{ for } 0 < \tau \le T_0. \tag{1.5}$$

We can reduce the above problem to one with fixed boundaries, by introducing the new space variable $x = s^{-1}(\tau)y$, the corresponding transformation of τ defined by

$$\frac{d\tau}{dt} = s^2(\tau)\ \tau(0) = 0, \tag{1.6}$$

lead for the function $u(x,t) = v(y,\tau)$ to the problem :

***Problem P^u*:** Find u such that

$$u_{xx} - u_t = x u_x(1,t) \text{ in } Q = \{(y,t) | x \in I, 0 < t \le T\},$$

$$\left.\begin{array}{l} u_x(0,t) = 0 \\ u(1,t) = 0 \end{array}\right\} \text{for } 0 < t \le T \quad ,$$

$$u(x,0) = g(x) \text{ for } x \in I,$$

$$\left.\begin{array}{l} \dfrac{ds}{dt} = -u_x(1,t) \\ s(0) = 1 \end{array}\right\} \text{for } 0 < t \le T, \tag{1.7}$$

hence $t = T$ corresponds to $\tau = T_0$ .The original Stefan problem is now is split into a non-linear PPDE initial boundary value problem ,for a fixed domain ,and two ordinary differential equations (1.6) and (1.7).

For two dimensional problem, the moving boundary has the form $y = s(x,t)$, and in the associated coordinate transformation the independent variables x and t are retained but y is replaced by

$$\zeta = \frac{y}{s(x,t)}, \tag{1.8}$$

thus for the one phase problem with the governing PPDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, 0 < x < 1,\ 0 < y < s(x,t).$$

with boundary condition are given on $x = 0$, $x = 1$, $y = 0$, and on $y = s(x,t)$ there are two condition, one specifying u and the other give the relation between the velocity of the moving boundary and the normal derivative of u at this boundary [1]. The transformation (1.8) lead the new equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial \zeta} + c\frac{\partial^2 u}{\partial \zeta^2} + e\frac{\partial u}{\partial x} \qquad 0 < x < 1,\ 0 < \zeta < 1, \tag{1.9}$$

where $b = -\frac{2y}{s^2}\frac{ds}{dx}$, $c = \frac{1}{s^2} + \frac{b^2}{4}$, $e = \frac{\partial^2}{\partial x^2}\left(\frac{y}{s}\right) + \frac{1}{s}\left(\frac{\partial y}{\partial t}\right)$ and the new boundary condition are obtained ,for example the condition $\frac{\partial u}{\partial x} = 0$ on $x = 0$ becomes

$$s\frac{\partial u}{\partial x} - \frac{y}{s} + \frac{\partial s}{\partial x}\frac{\partial u}{\partial \zeta} = 0 \quad \text{on} \quad x = 0.$$

The velocity of the moving boundary is specified to be $-\frac{\partial u}{\partial x}$, thus if $\zeta = 1$ in the new plane we have

$$\frac{\partial y}{\partial t} = -\frac{1}{s}\left(1 + \left(\frac{\partial s}{\partial x}\right)^2\right)\frac{\partial u}{\partial \zeta} \tag{1.10}$$

To find the numerical solution, we can use explicit, implicit, scheme such as forward difference in time with central difference in space.

2. Model Equation

Heat conduction through a one-dimensional region involving a phase change at a single point, $s(t)$, (ie. a boundary moving with time) is described by

$$\frac{\partial}{\partial x}\left(k_i \left.\frac{\partial u}{\partial x}\right|_i\right) = \rho_i c_i k_i \left.\frac{\partial u}{\partial x}\right|_i \tag{2.1}$$

where $i = 1$ for $0 < x < s(t)$ and $i = 2$ for $x \ge s(t)$, along with initial conditions defined at $t = 0$ and boundary conditions specified at $x = 0$ and $x = L$ or as $x \to \infty$ .At the phase change boundary, $x = s(t)$,

$$u_1 = u_2 = u_m \tag{2.2}$$

and

$$k_2 \frac{\partial u}{\partial x}\bigg|_{s(t)} - k_1 \frac{\partial u}{\partial x}\bigg|_{s(t)} = \rho_1 L \frac{ds(t)}{dt} \tag{2.3}$$

where u_m is the phase change temperature and L is the latent heat involved in the phase change, these equations plus the relevant initial and boundary conditions constitute the definition of a single boundary Stefan problem, and providing the solution for $u_1(x, t)$, $u_2(x, t)$, and $s(t)$.

In a solidification problem, for example, region 1 may represent that fraction of the body in the solid state with region 2 defining the liquid portion. In this case the latent heat L is negative and $s(t)$ is positive whilst for melting problems these signs are reversed, using this notation, a suitable test problem, involving freezing, has been described by Goodrich [7] and is outlined below

$$u_m = 0 \, ; u(\infty, t) = 2 \, ;$$

$$u(x,0) = 2 \quad (x \ge 0) \, ; u(0,t) = -4 \quad (t > 0)$$

$$k_i = k = 2 \times 10^5 \, ; c_i = c = 2.5 \times 10^7$$

$$\rho_i = \rho = 1 \, ; \; L = 100 \times 10^7$$

The analytic solution to this type of problem, [8], is given by

$$s(t) = 2\lambda(\kappa_1 t)^{1/2} \, ,$$

where $\kappa_i = \dfrac{k_i}{\rho_i c_i}$, $s(t)$ is the position of the boundary at time t , λ is a real constant, The exact solution of a point 25cm from the surface is shown as curve (a) in Figure (3.1). The characteristic "knee" as the temperature passes through the phase change

3. NUMERICAL SOLUTION

We will allow the phase to change and occur over a temperature range, this is a physically acceptable since many of the practical problems involving a phase change fall into this category,

Having what is called a 'mushy' region. In this case if the grid size step is constrained so that more than one grid point always falls into the phase change region then the rogue plateaus disappear. This is because the behavior of the grid point temperatures adjacent to a grid point neighboring and being approached by the phase change region, is always monotonically decreasing (until it enters the phase change region it self) . A number of papers have been published which effectively do this through the basic scheme [1],[2],[9]. If the initial temperature of the region is the phase change temperate then the above condition is always satisfied and an accurate numerical solution can be generated.

There are three main practical, methods to find the numerical solution, of the above problem, which effectively spread the phase change over a given region.

3.1-Szekely and Lee's Method

Considered the enthalpy *H(T)* of the material under consideration (ie. the sum of the sensible and latent heats) as a continuous function of temperature [6]. Thus, equation (2.1) may be rewritten as

$$\frac{\partial}{\partial x}\left(1\frac{\partial u}{\partial x}\right)=\rho\frac{dH}{du}\frac{\partial u}{\partial t}, \qquad (3.1)$$

with,

$$\frac{dH}{du}=\begin{cases} c & u\le u_m; u\ge u_2 \\ L/(u_2-u_1)\,f(u) & u_1<u<u_2 \end{cases}, \qquad (3.2)$$

where u_1 and u_2 define the region within which the phase change occurs. Note that equation (3.2) ensures that the heat gained on passing the phase change region is $L+c(u_2-u_1)$. If we use explicit finite difference approximation then equation (3.1) become

$$H_i^{j+1}=H_i^j+R\left(u_{i+1}^j-2u_i^j+u_{i-1}^j\right), \qquad (3.3)$$

where $R=k\lambda/\rho$ and dH/du is evaluated at each time step when k is constant. This problem was solved by a method developed by Szekely and Lees, [4], with using a mesh size $\Delta x=12.5cm$ and $\Delta t=3600\,sec$. Twenty distance steps were used and dH/du was found from a suitable finite difference representation of equation (3.2) with the half mushy range ε , taking the values 10^{-6} , 0.5 and 1.5^0C . The cooling curves for the point $x=25cm$ are shown in Figure (3.2). When $\varepsilon=10^{-6}$ the cooling curve is smooth and is similar to a problem whose latent heat is zero (i.e. no phase change). The curve for $\varepsilon=.5$ is a combination of jumps and plateauz, whereas the curve for $\varepsilon=1.5$ approaches the smoother behavior expected. From these results it would seem that the size of the mushy region strongly influences the stability of the numerical solution.

3.2-Meyers method

This method was developed by Meyer [7], where he use implicit scheme to find the numerical of the problem Meyer uses equation (2.1)with *H(u)* a piecewise continuous function given by

$$H(u)=\begin{cases} c_1 u & u \le u_m-\varepsilon \\ H(u_m-\varepsilon)+L(u-u_m+\varepsilon) & u_m-\varepsilon \le u \le u_m+\varepsilon \\ H(u_m+\varepsilon)+c_2(u-u_m+\varepsilon) & u \ge u_m+\varepsilon \end{cases}, \tag{3.4}$$

We note that the rise in H across the phase change region is L, so strictly speaking the function *H(u)* is not the enthalpy. Equation (2.1) may be approximated , using implicit finite difference approximation , by

$$R^*(H(u_i^j)-H(u_i^{j-1})-u_{i-1}^j+2u_i^j-u_{i-1}^j=0, \text{ where } R^*=\rho\Delta x^2/k\Delta t \tag{3.5}$$

.

equation (3.5) can be expressed in matrix form as

$$Au+\phi(u)=0 \tag{3.6}$$

where $u=(u_1,\ldots,.u_N)$ is the vector of nodal values, A is the matrix of nodal temperature coefficients and $\phi(u)$ is a vector with elements $R^*(H(u^j)-H(u^{j-1})$ .The above nonlinear system equations (3.6) ,is solved by using the successive relaxation iterative technique [3].

Meyer demonstrated the strength of his method by solving a two-dimensional problem involving a well insulated chamber with its initial temperature below the phase change temperature [5].

By Observed Meyer's problem the semi-plateau occurring in the temperature plot some time after the point has passed through the phase change. This semi-plateau may be directly associated with its neighboring points passing through the phase change, these observations call into question of the accuracy and stability of Meyer's method and can be seen as in the test problem using $\varepsilon=.01$, 1.5 . The results, illustrated in curves (a) and (b) of Figure(3.3), confirm that ε is a critical parameter in obtained stable solutions. Curve (c) in Figure(3.3) calculated for $\varepsilon=.01$ and $\Delta x=5$ shows only a , slight improvement with a smaller x-step.

We observe that, there still remains a demand for an accurate, reliable and flexible technique to describe phase changes which may occur either at a point or over a range.An algorithm is needed for calculating the position of moving boundary/region accurately , so we modified , and developed , an algorithm to find the numerical solution of equations (3.1)and (3.2) we will use HOC finite difference method to find the numerical solution of the above problem .

3.3-Reyadh and Saad method

We will develop the method in which we will use the HOC scheme developed in [10] to find the numerical solution ,where the truncation errors of compact difference operators for second derivatives is of order 4 that is

$$\left.\frac{d^2u}{dx^2}\right|_i=\delta_x^2 u_i-\frac{h^2}{12}\left.\frac{d^4u}{dx^4}\right|_{ii}+O(h^4)\,. \tag{3.7}$$

δ_x^2 in (3.7), is the standard central difference operator for second derivative of u .To find the forth derivatives we differentiate (3.1) with respect to x

$$\frac{\partial^3 u}{\partial x^3} = \rho \frac{dH}{du} \frac{\partial^2 u}{\partial t \partial x},$$

$$\frac{\partial^4 u}{\partial x^4} = \rho \frac{dH}{du} \frac{\partial^3 u}{\partial t \partial^2 x},$$

And using the above equation in (3.7) we have

$$\left.\frac{d^2 u}{dx^2}\right|_i = \delta_x^2 u_i - \frac{h^2}{12} \rho \frac{dH}{du} \frac{\partial^3 u}{\partial t \partial^2 x} + O(h^4).$$

Thus the resulting high-order scheme of (3.1)

$$\delta_x^2 u_i - \frac{h^2}{12} \rho \frac{dH}{du} \delta_t^+ \delta_x^2 u = \rho \frac{dH}{du} \delta_t^+, \qquad (3.8)$$

where is δ_t^+ the standard forward difference operator for the first derivative of u with respect to the time t and the operator $\delta_t^+ \delta_x^2$ given by

$$\delta_t^+ \delta_x^2 = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} - (u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{kh^2}$$

In our method we use the backward Euler finite difference operator with the continuous function of temperature H defined by Szekely and Lee's in equation (3.2) i.e.

$$\frac{dH}{du} = \begin{cases} c & u \le u_m ; u \ge u_2 \\ L/(u_2 - u_1) f(u) & u_1 < u < u_2 \end{cases},$$

so that the backward Euler HOC finite difference form of (3.1) has the form

$$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} - \frac{h^2}{12} \rho \frac{dH}{du} \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} - (u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{kh^2} \right) = \rho \frac{dH}{du} \frac{u_i^{n+1} - u_i^n}{k},$$

The above scheme can be simplified to be

$$(A+B)u_{i-1}^n + (-2B - 2A + C)u_i^n + (A+B)u_{i+1}^n - Au_{i-1}^{n+1} + (2A - C)u_i^{n+1} - Au_{i+1}^{n+1} = 0 \qquad (3.8)$$

where

$$A = \frac{1}{12k} \rho \frac{dH}{du}, B = \frac{1}{h^2} \text{ and } C = \frac{\rho}{h^2} \frac{dH}{du}. \qquad (3.9)$$

The derivative dH/du was approximated by using forward, backward ,finite difference operators and thus equation (3.2) with the half mushy range ε, taking the values 10^{-6}, 0.5 and $1.5^0 C$.

In Figure(3.4) we plot the solution of equation (3.8) and we can see the improvement in the solution, aseptically at the mushy range $\varepsilon = 10^{-6}$, where the cooling curve is smooth and explain the behavior of the solution better than the solution of Szekely using the same half mushy range ε , we also use the forward Euler finite difference, implicit method, to find the numerical solution to equation (3.1)with the function *H(u)* defined by Meyer in equation (3.4) i.e.

$$H(u) = \begin{cases} c_1 u & u \le u_m - \varepsilon \\ H(u_m - \varepsilon) + L(u - u_m + \varepsilon) & u_m - \varepsilon \le u \le u_m + \varepsilon \\ H(u_m + \varepsilon) + c_2(u - u_m + \varepsilon) & u \ge u_m + \varepsilon \end{cases} .$$

From equation (3.8) and if we use the implicit HOC finite difference operator, equation (3.1) may approximated in implicit form as

$$\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} - \frac{h^2}{12}\rho\frac{dH}{du}\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} - (u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n})}{kh^2}\right) = \rho\frac{dH}{du}\ \frac{u_i^{n+1} - u_i^{n}}{k}, \quad (3.10)$$

this can be simplified to be

$$Au_{i-1}^{n} + (-2A + C)u_i^{n} + (A)u_{i+1}^{n} + (B - A)u_{i-1}^{n+1} + (-2B + 2A - C)u_i^{n+1} - (B - A)u_{i+1}^{n+1} = 0, \quad (3.11)$$

where A, B and C are given in (3.9)

The system of non-linear equation (3.11) can solved by using safe guarding Newton method ,[5] ,and the numerical solution of (3.11) is plotted in Figure (3.5), We can conclude that the smooth behavior of cooling curve at the mushy range $\varepsilon = 0.1$, is more accurate than the cooling curve given by Meyer method at the same mushy range and one can notice the closeness between the exact and the numerical solution

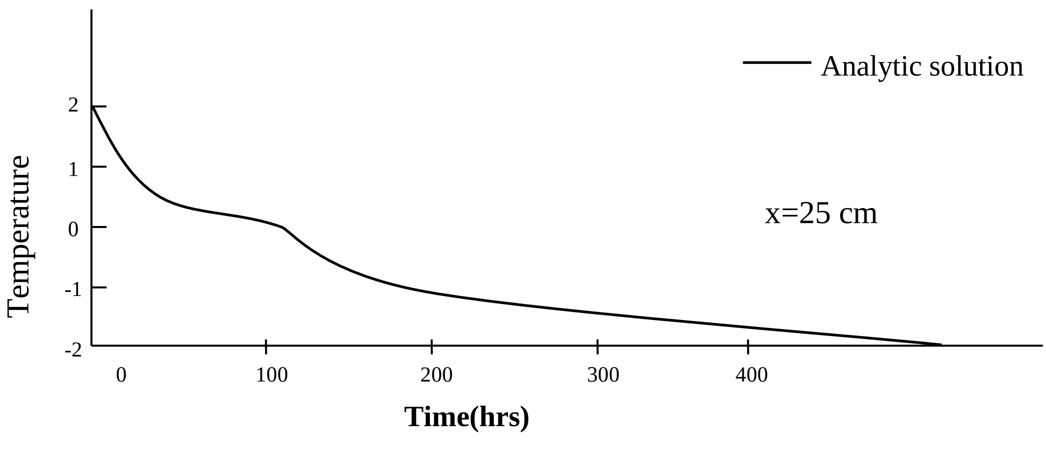

Figure 3.1 The exact and numerical solution of a point 25cm of test problem

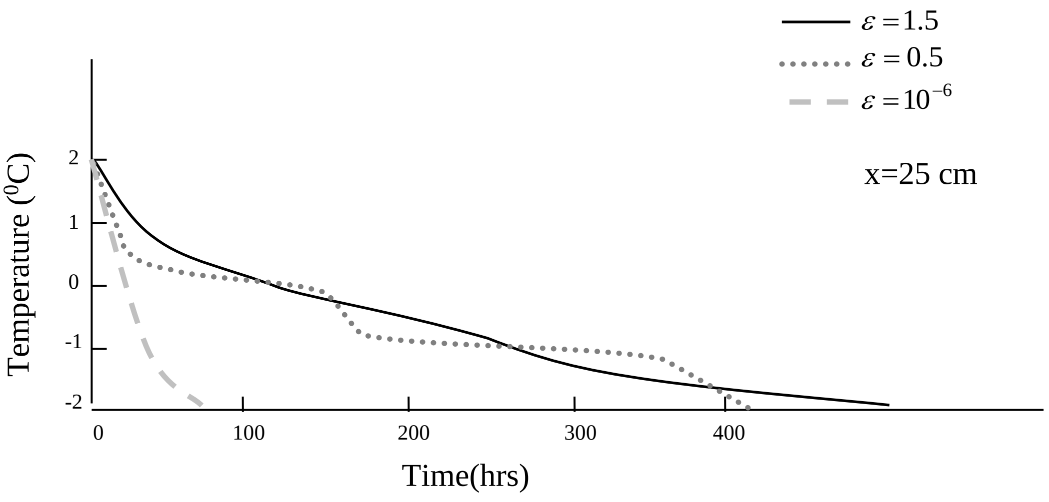

Figure 3.2 The numerical solution of a point 25cm by Szelely Method

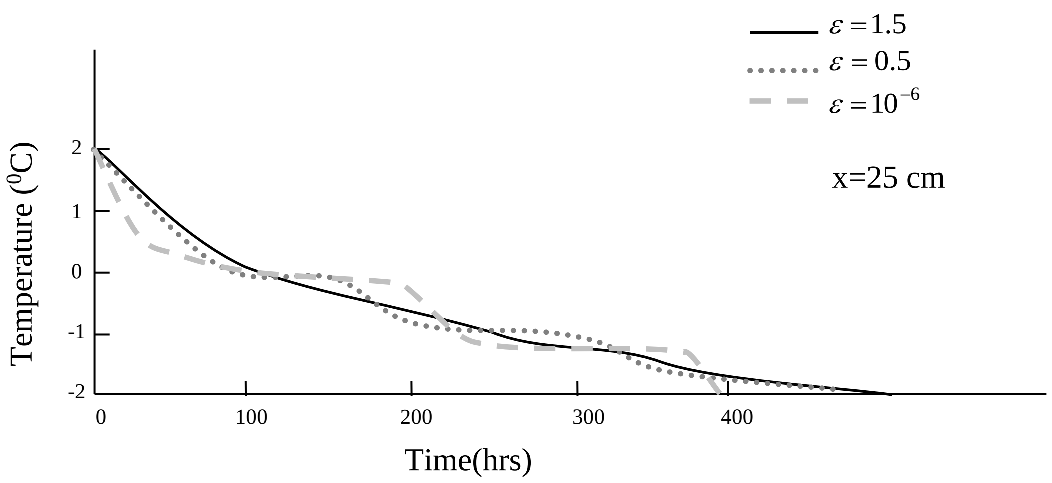

Figure 3.3 The numerical solution of a point 25cm by Meyer Method

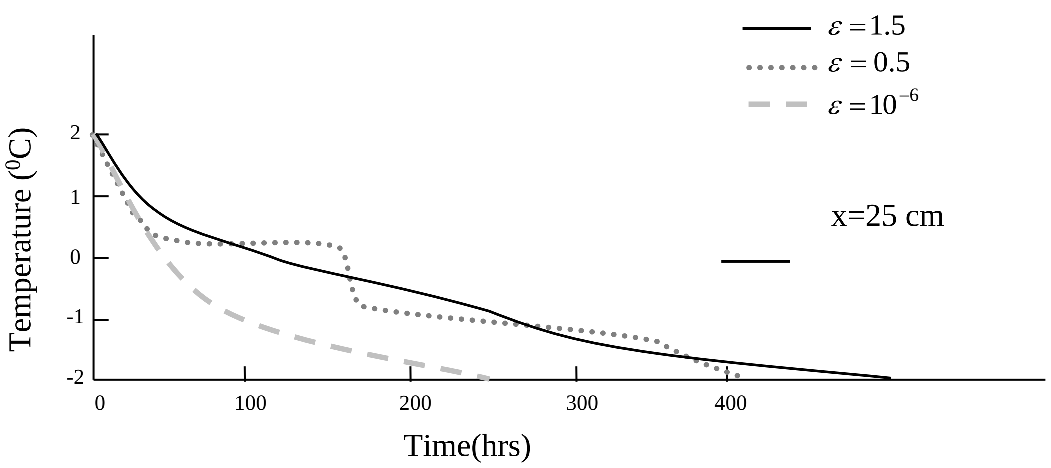

Figure 3.4 The numerical solution of a point 25cm using the backward HOC finite difference method

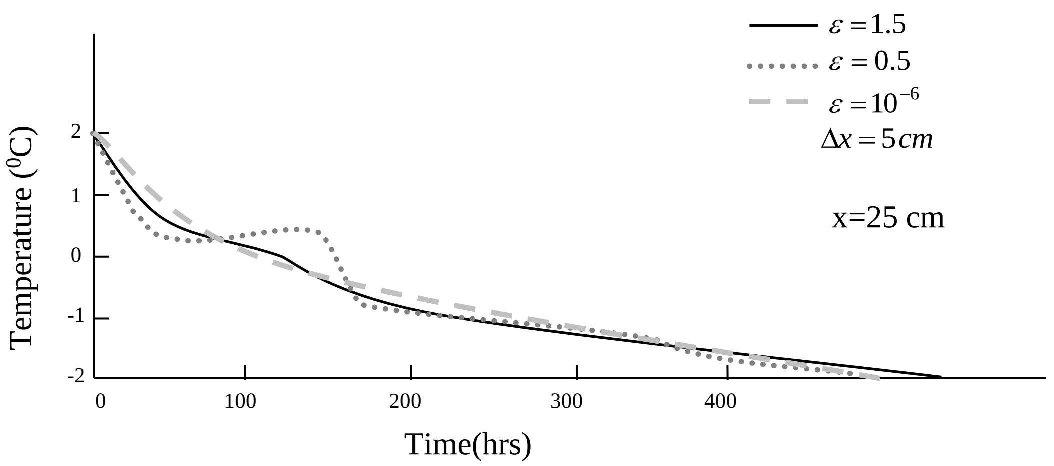

Figure 3.5 The numerical solution of a point 25cm using the forward HOC finite difference method

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