On feebly continuous functions and feebly compact space

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Abstract :

The main purpose of this paper is to study a new kinds of continuous functions called feebly continuous function , f^{*}eebly continuous function and f^{**}eebly continuous functions and study some of their characterizations and proving number of theorems , corollaries , propositions , remarks , examples and their relationships with continuous functions .Also we will introduce a new concept of compact spaces called feebly–compact spaces (f- compact space) and study the relationship of it with usual compact spaces .

1- Introduction :-

The concepts of semi open (*s- open*) and semi closed (*s-closed*) sets are introduced by N.Levine in (1963)[6] .He defined a set A in a topological space X to be s-open if for some set G , G $\subseteq$ A $\subseteq$ cl (G) where cl(G) denote the colsure of G in X. A set is s-closed if it's complement is s-open .

In 1971 S.G .Crossiey and S.K.Hildebrand introduced the concept semi closure and they defined it , the semi closure of a set A in a topological space X is the smallest semi closed (s-closed) set containing A [3] and denote it by scl(A). In fact scl(A) is the intersection of all the semi closed sets containingA,A$\subseteq$scl(A)$\subseteq$cl(A),scl(Scl(A))=scl(A).

Feebly closed and feebly open set were introduced by Maheswari and Tapi [7] in 1978. A set A in a topological space X is feebly open (*f-open*) set if there exist an open set G such that G $\subseteq$ A $\subseteq$ scl (G) . Every open set is feebly open set but the converse may be false . A set is feebly closed if its complement is feebly open. Every closed set is feebly closed set but the converse may not be true[7]. A function $f : X \to Y$ is called feebly open (closed) *f-open (f-closed)* function if the image of any open(closed) set in X is feebly open(closed) *f-open (f-closed)* set in Y, and it is called f^**-open* (f^**-closed)*function,if the image of each *f-open(f-closed)* set in X is open (colsed) set in Y, and called f^{**}*-open*(f^{**}*-closed)* function , if the image of each *f-open (f-closed)* set in X is *f-open (f-closed)* set in Y[1].

2- Preliminaries :-

The following are known for any mathematician but they are written because we need them.

Defintion (2-1) [4]

Let X , Y be two topological spaces , A function $f : X \to Y$ is called continuous at $x_0 \in X$ iff for each open set H in Y containing $f(x_0)$, $\exists$ an open set G in X containing x_0 such that $f(G) \subset H$.
A function is continuous iff it is continuous at each points of X.

Defintion (2-2) [4]

Let X be any set and $T=\{A \in P(X) : A^c$ is finite $\}$, then (X,T) is called cofinite topological space ,and denoted by T_c .

Remark (2-3) [5]

A function $f : X \to Y$ is called *mapping (maps)* if f is continuous.

Theorem (2-4) [5]

Let X , Y be two topological spaces then $f : X \to Y$ is cotinuous iff the inverse image of every open set in Y is open set in X .

3- Feebly, F^*eebly and F^{**}eebly continuous Functions

In this section we introduce new kinds of continuous functions called $f$eebly continuous function, f^*eebly continuous function , and f^{**}eebly continuous function, and we study their properties and the relationships among them.

(a) **Feebly continuous Functions**.

Here we give the definition of feebly continous function.

Definition(3-1)

Let X , Y be two topological spaces . A function $f : X \to Y$ is called feebly continuous (f-continuous) function if the inverse image of every closed(open) set in Y is f-closed (f- open) set in X , denoted by f- continuous mapping .

Example (3-2)

Let X ,Y be two topological spaces on X , and (X,T_1) be indiscrete topology on X , (Y,T_2) be any topology on Y , $f : X\to Y$ be any function, then f is f- continuous function .

proposition : (3-3)

Every continuous function is f-continuous function .

Proof :

Let X , Y be two topological speace and $f : X \to Y$ be continuous function, let A be any open set in Y .

$\because f$ is continous function, then $f^{-1}(A)$ is open set in X.

$\therefore f^{-1}(A)$ is feebly open set in X,

then f is f- continuous function.

□

Remark : (3-4).

The inverse of proposition (3-3) not necessary true as in the following example :

Let X={a,b,c} , T_1 is indiscrete topology on X ,

T_2 = { ϕ , X , {a},{b},{a,b}} be another topology on X,

and $g:(X,T_1)\to(X,T_2)$ defind as : $g(x)=x$.

Then g is f- continuous function,because the inverse image for each open (closed)set in T_2 is feebly open (closed) in T_1.

but it is not continuous function because the set {c} is closed in T_2 , but $g^{-1}\{c\}=\{c\}$ is not closed set in T_1.

(b) ***F^**eebly continuous Functions**:

Defintion 3-5

Let X,Y be two topological spaces .A function $f :X\to Y$ is called f^*eebly continuous function(denoted by f^*- continuous) if the inverse image of every feebly open(closed) set in Y is open (closed) set in X .

Example (3-6)

Let X,Y be two topological spaces and

T_1 is discrete topology on X .

T_2 be any topology on Y .

then $f :(X,T_1)\to (Y ,T_2)$ is f^*- continuous function .

Proposition (3-7)

Every f^*- continuous function is continuous .

Proof :-

Let X,Y be two topological speaces and $f :X\to Y$ be f^*- continuous function .

Let A be any open set in Y, then A is feebly open set in Y,

$\because$ f is f^*- continuous function .
$\therefore$ $f^{-1}(A)$ is open set in X.
$\therefore f$ is continuous function.

□

Remark (3-8)

The inverse of proposition (3-7) is not necessery true as in the following example:

Let T_u be usual topology on R, T_i be indiscret topology on R ,
$f:(R, T_u) \to (R, T_i)$ be a function defined by :
$f(x)=2x \quad \forall\ x \in R$ then

i- f is continuous function because $f^{-1}(R) = R$ and $f^{-1}(\phi) = \phi$ are open (closed)sets in (R, T_u) .

ii- f is not f- continuous function because if $G=\{2\}$ is feebly open set in T_i but $f^{-1}(G) = \{1\}$ is not open set in (R, T_u) .

(c) ***F*** ****eebly continuous Functions**:

Defintion (3-9)

Let X ,Y be two topologyical spaces , A function $f:X \to Y$ is called f^{**}eebly continuous function (denoted by f^{**}-continuous) if the inverse image of every feebly open (closed) set in Y is feebly open(feebly closed) set in X .

Example (3-10)

In example (3-6) clearly f is f^{**}- continuous function because the inverse image of any feebly open (closed) set in (Y,T_2) is feebly open(closed) set in (X,T_1) .

proposition(3-11)

Every f^*-continuous function is f-continuous .

Proof :-

By the same method in proving propositions (3-3) and (3-7).

□

Remark (3-12) :

The inverse of proposition (3-11) is not necessary true as in the following example :
in remark(3-4) , hence g is f-continuous function because g^{-1} for each closed set in T_2 is feebly closed set in T_1, and it is not f^*-continuous function because {c} is feebly closed set in T_2 but g^{-1} {c} = {c} is not closed set in T_1.

proposition (3-13)

Every f^{*}- continuous function(mapping) is f^{**}- continuous function .

Proof :-

By the same method in proveing propositions (3-3) and(3-7).

□

Remark (3-14)

The inverse of proposition (3-13) is not necessary true as in the following example .

Let T_u be usual topology on R, T_i be indiscret topology on R,

$f: (R, T_i) \to (R, T_u)$ be a function defind by :

$f(x)= x \quad \forall\ x \in R$ then

i- f is f^{**}-continuous function because the inverse image of any feebly closed set in (R,T_u) is feebly closed set in (R,T_i).

ii- f is not f^{*}-continuous function because if $G = (a,b)$ is feebly open set in (R,T_u) but $f^{-1}(G) = (a,b)$ is not open set in (R, T_i).

proposition (3-15)

Every f^{**}-continuous function is f-continuous function.

Remark(3-16)

The inverse of proposition (3-15) is not necessary true as in the following example:

Let X={a,b,c} , $T_1 = \{\phi, X, \{a\}, \{a,b\}\}$ be topology on X .

and$T_2 = \{\phi, X\}$ is the indiscrete topology on X .

$f: (X,T_1) \to (X,T_2)$ defined by $f(x) = x$.

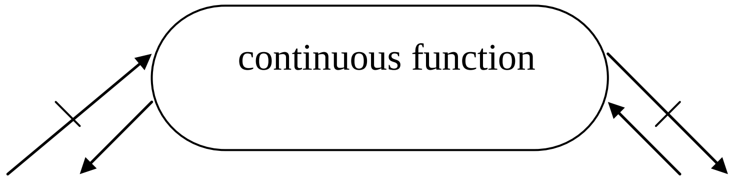
then :

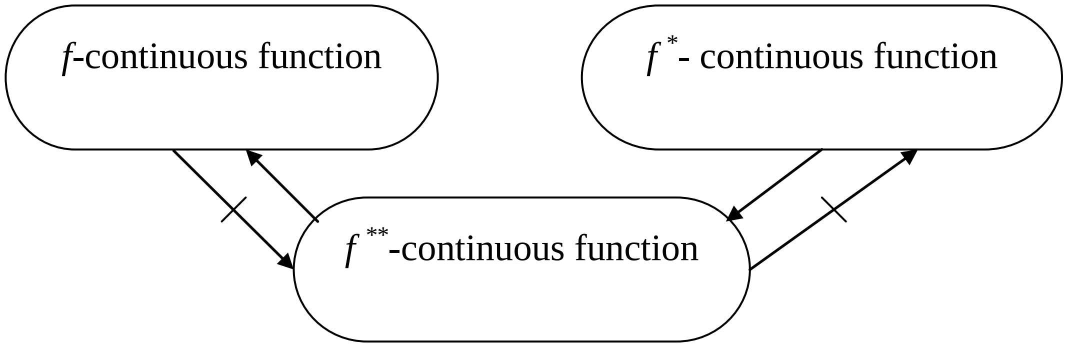
i- f is f-continuous function because the inverse image of any closed set in T_2 is feebly closed set in T_1

i.e $f^{-1}(\phi)=\phi$, $f^{-1}(X)=X$ are feebly closed set in T_1.

ii- f is not f^{**}-continuous function because {b} is feebly closed set in T_2 but $f^{-1}\{b\} = \{b\}$ is not feebly closed in T_1 .

The following diagram shows the relation among continuous function,f-continuous function,f^{*}-continuous and f^{**}- continuous function.





4- The main Results.

feebly compact space (f-compact space)

In this section we define a new concept of compactness called feebly compact space (f-compact space) and we will study it's relationship with usual compact space .

Definition (4-1):

Let (X,T) be a topolegical space , $A \subseteq X$.
A family $\{G_i : i \in \Delta\}$ of subsets of X is cover for A iff $A \subseteq \bigcup_{i \in \Delta} G_\Delta$, and it's called open cover if the elements of the cover are open.

Definition (4-2):

Let a family $\{G_i : i \in \Delta\}$be an open cover of a set A , then if their exist finite subset Δ_o of Δ such that
$A \subseteq G = \bigcup_{i \in \Delta_o} \{G_i : i \in \Delta_o\}$ then G is called finite subcover of A.

Definition (4-3):

Let *(X,T)* be a topological space , X is called compact space iff every open cover of X has finite subcover.

Remark(4-4):

Let *(X,T)* be a topological space , $A \subseteq X$ then A is called compact subset on X , if every open cover of A has finite subcover.

Example (4-5):

Let (N,T_c) be cofinite topology on N, then (N,T_c) is compact space because :
If $U = \{U_i : i \in \Delta\}$ any open cover of N , let U_{io} any set in U then $N\text{-}U_{io}$ is finite set , let $N\text{-}U_{io} = \{n_1, n_2, \ldots, n_p\}$ then their exist finite sets $U_{i1}, U_{i2}, \ldots, U_{ip}$ hence $nk \in U_{ik}$, $k = 1,2,\ldots,p$ then the family $\{ U_{i1}, U_{i2}, \ldots, U_{ip} \}$ is finite subcover of N,

Hence *(N,T_c)* is compact space.

Now we present the definition of feebly compact (f-compact) space after introducing the concept of feebly open cover (f – open cover).

Definition (4-6):
A family $\{V_i : i \in \Delta\}$ of feebly open sets (f-open sets) in *(X,T)* is called feebly open cover (f- open cover) of A iff $A \subseteq \bigcup_{i\in\Delta} V_i$.

Definition (4-7):
A topological space *(X,T)* is called feebly compact (f-compact) space iff every f-open cover of X has finite subcover .

Remark (4-8):
In a topological space *(X,T)* , A $\subseteq$ X then A is called feebly compact subset (f-compact subset) on X , if every f-open cover of a set A in X has finite subcover.

Example (4-9):
The topological space in example (4-5) is f-compact space because the f-open set in *N* are ϕ, N only.

Now we will study the relationship betweem compact and f-compact spaces and we will find that they are independent concepts as in the following examples:

Example (4-10):
Let (R,T_i) be the indiscrete topology on *R* , (R,T_i) is compact space but it is not f-compact space , because every subset of R is f-open set then the family of singlton set in *R* *W={x}* $\forall x \in R$ is f-open cover of *R* but *W* hasn't finite subcover.

Example (4-11):
Let *(R,P)* be a topological space defind on R as :
P= $\{\phi\} \cup \{A \subseteq R : 0 \in A\}$. Then *(R,P)* is not compact space because the open cover *G={(-n,n):n=1,2,3,...}* haven't finite subcover of R , but *(R,P)* is f-compact space because the f-open set in R are ϕ, R only.

Theorem (4-12): [2]
Let $f : X \to Y$ be continuous and onto function, if X is compact space then Y is compact space .

Now we will study the f-compact propertes

Theorem (4-13):
Let $f : X \to Y$ be f -continuous and onto function if X is f-compact space then Y is compact space .

Proof : Let $\{K_i : i \in \Delta\}$ is open cover of Y.
$\because f$ is f- continuous function (mapping).
$\therefore \{f^{-1}(K_i) : i \in \Delta\}$ is f-open cover of X.
$\because X$ is f-compact space.
$\therefore X$ has finite subcover $\{f^{-1}(K_1), f^{-1}(K_2),..., f^{-1}(K_n)\}$
$\because f$ is onto function (mapping) .
$\therefore f(f^{-1}(K_i) = K_i, \forall i \in \Delta$
$\therefore \{K_1, K_2,..., K_n\}$ is finite subcover of Y.
$\therefore Y$ is compact space.

□

Theorem (4-14):
Let $f : X \to Y$ be f^*-continuous and onto function , if X is f-compact space then Y is f-compact space.
Proof :
Let $\{G_i : i \in \Delta\}$ is f-open cover of Y ,
$\because$ f is f^*-continuous function,
then $\{f^{-1}(G_i) : i \in \Delta\}$ is open cover of X,
then $\{f^{-1}(G_i) : i \in \Delta\}$ is f-open cover of X,
(every open set is f-open set)[def 2-3]
$\because X$ is f-compact space, $\therefore X$ has finite subcover
$\{f^{-1}(G_1), f^{-1}(G_2),..., f^{-1}(G_n)\}$
$\because f$ is onto function then $f(f^{-1}(G_i)) = G_i : \forall i \in \Delta$
$\therefore$ $\{G_1, G_{2,},,, G_n\}$ is f-open cover of Y .
$\therefore$ Y is f- compact space.

□

Theorem (4-15):

Let $f : X \to Y$ be f^{**}-continuous and onto function , if X is f-compact space then Y is f-compact space.

Proof :

Let $\{G_i : i \in \Delta\}$ is f-open cover of Y ,

$\because$ f is f^{**}-continuous function,

then $\{f^{-1}(G_i) : i \in \Delta\}$ is f-open cover of X,

$\because X$ is f-compact space, $\therefore X$ has finite subcover

$\{f^{-1}(G_1), f^{-1}(G_2),..., f^{-1}(G_n)\}$

$\because f$ is onto function then $f(f^{-1}(G_i)) = G_i : \forall i \in \Delta$

$\therefore$ $\{G_1, G_{2,},,, G_n\}$ is f-open cover of Y .

$\therefore$ Y is f- compact space.

□

Remarks (4-16):

1- Let X, Y be two topological spaces and let $f : X \to Y$ be function , if A is compact subset of *X* then:

i- *f(A)* is compact subset of *Y*, if *f* is open function and bijective function .

ii- *f(A)* is *f*- compact subset of *Y*, if *f* is *f*-open function and bijective function .

2- Let X, Y be two topological spaces and let $f : X \to Y$ be function , if A is *f*- compact subset of *X* then:

i- *f(A)* is compact subset of *Y*, if *f* is f^*-open and bijective function .

ii- *f(A)* is *f*- compact subset of *Y*, if *f* is f^{**}-open and bijective function .

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حول الدوال المستمرة الواهنة والفضاءات المرصوصة الواهنة

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المستلخص:

سيتم في هذا العمل دراسة أنماط جديدة من الدوال المستمرة تسمى الدوال المستمرة الواهنة *(feebly continuous)* ، الدوال المستمرة

الواهنــــة* (*f*eebly continuous*) والـــدوال المســـتمرة الواهنـة** (*f**eebly continuous*) ودراسة بعض خواص هذه الأنمـاط وبرهنـة عدد من المبرهنـات والنتـائج والقضـايا الخاصـة بـذلك واعطـاء عـدد مـن الملاحظـات والأمثلـة ودراسـة علاقتهـا مـع الـدوال المستمرة . وكذلك تقديم صيغة جديدة من صيغ الفضـاءات المرصوصـة تسـمى الفضـاءات المرصوصـة الواهنـة (*f-compact space*) ودراسة علاقتها مع الفضاءات المرصوصة الاعتيادية وبرهنة بعض المبرهنات والملاحظات والأمثلة التي تبين تلك العلاقة.