Probabilities Maps In Homogeneous Combat

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الخرائط الاحتمالية في المعارك المتجانسة

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Abstract

Lanchester- type attrition models refer to the sct of differential equation models that describe changes, over time, in the force levels of combatants and other significant variables that describe the combat process. Lanchester-type models express casualties\ attrition in terms of force size, and other associated variables and how they change over time Lanchester differential equation models have gained importance through their ability to provide insight into the dynamics of combat.

This paper focused on derivative Mathematics of the Lanchester Square Law (aimed fire) and Mathematics of the Lanchester Linear Law (area fire) , we calculated the rate and attrition equations by The Kolmogorov Differential equations and markov transition equations finally we offered Discussion & Conclusion.

CHAPTER -1-

LANCHESTER- TYPER DIFFERENTIAL ATTRITION MODELS

1-1 Introduction:

Lanchester- type attrition models refer to the sct of differential equation models that describe changes, over time, in the force levels of combatants and other significant variables that describe the combat process. Lanchester-type models express casualties\ attrition in terms of force size, and other associated variables and how they change over time. They may be simple models with closed form solutions capable of being solved through simple mathematics or they may be large, highly complex models requiring a variety of analytical and simulation techniques. Such models are used to answer such basic questions as who wins the battle or more complex operational questions pertaining to force mix or tactics.

Lanchester differential equation models have gained importance through their ability to provide insight into the dynamics of combat and their applicability to almost the entire hierarchy of combat operations (e.g. battalion through theater-level). In cases where simple models are utilized, explicit analytical forms may de derived and answers readily provided to client\ user. Further, these differential equation models provide a basis for developing quantitative insights into combat dynamics. The simple equations form the base for model enrichment that provides the means to simulate combat and address more critical operational problems.

While there exists a wide variety of Lanchester-type differential models based on size and complexity, there are several underlying factors that appear common to the model development process. These concepts are:

- attrition to a force is function of force size and other associated parameters (i.e. casualty ratc = f (force size; other possible parameters)).
- force size is a function of time, and the continuous real time variables x(t), y(t) and t are approximations to the discrete combat units a real force.
- if we consider two opposing forces X,Y and let

x(t)= size of the X force as a function of time
y(t)=size of the Y force as a function of time

the casualty rates can be written as a simple pair of differential equations

$$\frac{dx}{dt} = f(x, y) \qquad \frac{dy}{dt} = f(x, y)$$

- ❖ The solution to any such system of differential equations is pair a of functions giving x(t) and y(t) as a function of time.

1.2 Lanchesters Original Models

Origins of the Lanchester Models

In 1914 F.W. Lanchester, a British engineer and inventor, formulated two differential models for attrition under specific conditions of war. His purpose was to quantitatively justifying the principle of concentration of forces under the then conditions of modem warfarc. Lanchester hypothesized that in ((ancient warfare)), a battle was simply a collection of one –on -one ducls, with the casualty rate being independent of the number of units on the opposing side. Under "modern" conditions, he contended that the firepower\ lethality of weapons widely dispersed across the battlefield can be concentrated on surviving targets and a many- against –one situation could exist. Therefore, the casualty rates should be proportional to the size of the opposing force. Lanchester formulated some models based on ordinary differential equations to translate these hypotheses into mathematical terms.

Conditions of Ancient Warfare

Based on the hypothesized model for one- on –one duels, Lanchester argued that two forces of equal strength and fighting ability should intuitively be expected to lose about the same number of men. Further, under this one- on –one condition, any forces not engaged with an opponent must wait until an enemy soldier became available before joining combat. This implies that regardless of how large the X force is, it cannot engage the opposing Y force with more men than Y puts forth on the battlefield. Therefore under the condition of((ancient warfarc)) there should be no advantage in concentrating forces.

While never explicitly formulated, Lanchester`s ancient warfare equations reflect a combat attrition process where attrition rates are independent of force size; that is

$$\frac{dx}{dt} = -a \qquad \text{and} \qquad \frac{dy}{dt} = -b \qquad (1.2.1)$$

The individual X unit is superior to the individual Y unit if and only if b>a. Both sides decrease gradually in any case until one or the other becomes 0, at which point battle stops and 2.2.1 no longer holds. The relationship between x and y can be found from

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-b}{-a}=\frac{b}{a} \qquad (1.2.2)$$

Since the slope of y with respect to x is constant, x and y must at all times be related by

$$b[x_0-x]=a[y_0-y], \qquad (1.2.3)$$

Where x_0 and y_0 the initial values.

Conditions of Modern Warfare:

As previously noted, Lanchester defined the principal condition for modern warfare as the ability of firers to engage a single target. He based this condition on the advent of modern weapons that allowed multiple engagement possibilities and concentration of fires form weapons widely dispersed on the battlefield.

Considering the nature of modern weapons and how the concentration of fires could be achieved, Lanchester examined two general cases of combat, aimed fire and area fire. The first, aimed fire, assumes that individual targets are identified and attacked by any number of opposing systems\ fires. The second case, area fire, considers the situation where a force concentrates its fires over a general area occupied by the enemy and not at any particular enemy target.

Under aimed fire conditions, Lanchester stated that the attrition rate of x depends on how many y`s are shooting at him, and likewise for y. In mathematical terms.

$$\frac{dx}{dt}=-ay \qquad \frac{dy}{dt}=-bx \qquad (1.2.4)$$

Here a is an attrition rate coefficients expressed in terms of (X casualties)/ (Y firer) x (time), and similarly for b.

As will be shown below, it follows from 2.2.4 that x and y are related by

$$b[x_0^2-x^2]=a[y_0^2-y^2] \qquad (1.2.5)$$

Through the use of Lanchester- type combat models, it is possible to answer a variety of questions about combat between two forces.

1. Who will win the battle; or which force will be annihilated?
2. What force ratio is required to guarantee victory?

3. How many survivors will the winner have?
4. How long will the battle last?
5. How do the force levels change over time?
6. How do changes in the parameters {e.g. initial force levels (x_0 and y_0) or attrition coefficients (a and b)} affect the outcome of the battle?
7. Is concentration of forces a good tactic?

While the terms found in these questions are subject to various interpretations, more specific questions can be answered based on the complexity of the model and number of parameters incorporated. As additional parameters are added to model, more questions may be posed. However, for our purpose, discussion will be limited to how Lanchester-type models are developed and how they yield answers to seven basic questions listed above.

1.3 Mathematics of the Lanchester Square Law (aimed fire):

Lanchester originally hypothesized that combat between two homogenous forces under the conditions of modem warfare could be modeled as:

$$\frac{dx}{dy} = -ay \quad \text{where } x(0) = x_0$$

$$\frac{dy}{dt} = -bx \quad \text{where } x(0) = y_0$$

The equations hold only as long as both x(t) and y(t) are positive. Battle stops when either number becomes zero, if not before. Based on hypothesized differential equations, it is` possible to derive the equations that will allow us to determine who wins the battle, force size and time to battle termination.

Derivation of the State Equation:

Using the equations for force casualty rates, it is possible to derive an expression for the instantaneous casualty- exchange ratio as follows:

$$\frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{ay}{bx} = \frac{dx}{dy}$$

Separating the variables,

bx dx = ay dy

And integrating both sides, we discover that $bx^2 + c_1 = ay^2 + c_2$. Given the initial conditions, the constants must be such that at all times.

$$b(x_0^2 - x^2) = a(y_0^2 - y^2) \qquad (1.3.1)$$

Therefore given a value for either x or y, it is possible to solver for the other. However, it is important to note that we do not get any information about when any particular force level is achieved.

Force Levels as a Function of Time:

The pair of ordinary differential equations that determines x(t) and y(t) can be solved using standard methods. The solution is

$$x(t) = \frac{1}{2}\left(\left(x_0 - \sqrt{\frac{a}{b}}y_0\right)e^{\sqrt{ab}t} + \left(x_0 + \sqrt{\frac{a}{b}}y_0\right)e^{-\sqrt{ab}t}\right) \qquad (1.3.2)$$

$$y(t) = \frac{1}{2}\left(\left(y_0 - \sqrt{\frac{b}{a}}x_0\right)e^{\sqrt{ab}t} + \left(y_0 + \sqrt{\frac{b}{a}}x_0\right)e^{-\sqrt{ab}t}\right) \qquad (1.3.3)$$

The verity of these equations can be established by observing that x(0) and y(0) have the required values. And that the two differential equations are satisfied. Of course it should be understood that 2.3.2 and 2.3.3 hold only as long as both x(t) and y(t) are nonnegative.

Battle Outcome and Duration

To determine who will win the battle it is necessary to specify some condition that will cause the battle to terminate. Assume that the x-side will surrender or break off fighting in some other way if x(t) ever shrinks to x_{BP}, where of course $x_{BP} < x_0$, and similarly for the y- side and y_{BP}. At the terminal time, either x(t) $=x_{BP}$ and y(t)> y_{BP}, in which case the y-side is the winner, or y(t)- y_{BP}, x(t)>x_{BP}, and the x- side is the winner.

Case Y wins:

Since the X loses, the number of y-survivors y_f can be obtained by solving 1.3.1 with x=x_{BP}.

$$b(x_0^2 - x_{BP}^2) = a(y_0^2 - y_f^2)$$

$$Y_f = \sqrt{y_0^2 - \frac{b}{a}(x_0^2 - x_{BP}^2)}, \qquad (1.3.4)$$

Assuming that $y_f>y_{BP}$. The criterion for this be true; that is, the criterion for the y-side to win the battle, is

$$b(x_0^2 - x_{BP}^2) < a(y_0^2 - y_{BP}^2) \qquad (1.3.5)$$

The left and right-hand sides of 1.3.5 might be called the ((fighting strengths)) of the two sides, since the comparison determines the winner. Note that the number of participants on each side is squared, whereas the firepower rate coefficient is not, hence the term (Square Law). In Square Law battle, it is more important to have lots of units than it is have powerful units. Intuitively, adding one more unit to a square law battle serves two purposes: it fires at the enemy, and in addition it dilutes the enemy`s fire against existing units.

Increasing a firepower rate coefficient only serves the first purpose.

The length of the battle can be determined by solving 1.3.2 for t when $x(t)=x_{BP}$ Let z= exp($-t\sqrt{ab}$). Since exp($t\sqrt{ab}$) = 1\z, 1.3.2 is a quadratic equation in z. The only solution for which 0<z≤1 is

$$z = \frac{\sqrt{b(x_{BP}^2 - x_0^2) + ay_0^2} + \sqrt{bx_{BP}}}{\sqrt{bx_0} + \sqrt{ay_0}} \qquad (1.3.6)$$

The time t at which x(t) is x_{BP} is therefore

$$t = -\ln(z)/\sqrt{ab} \qquad (1.3.7)$$

For example, suppose a=01/day,b= .02/day, x_0= 20, and y_0= 40, with x_{BP}= y_{BP}=0. Then z= .4142,t =62.32 days and y_f = 28.28. In spite of having inferior units (a<b), Y wins with mots of his forces intact.

Casa X wins:

If $b(x_0^2 - x_{BP}^2) > a(y_0^2 - y_{BP}^2)$ (1.3.8)

Then y(t) will become y_{BP} before $x(t)=x_{BP}$; that is, X wins. The number of x- survivors is

$$x_f = \sqrt{x_0^2 - \frac{a}{b}(y_0^2 - y_{BP}^2)}. \qquad (1.3.9)$$

Solving the quadratic equation 1.3.3 with y(t) = y_{BP} for z as above, the solution is

$$z = \frac{\sqrt{a(y_{BP}^2 - y_0^2) + bx_0^2} + \sqrt{ay_{BP}}}{\sqrt{bx_0} + \sqrt{ay_0}}. \qquad (1.3.10)$$

With 1.3.7 still determining the time of battle termination.

1.4 Mathematics of the Lanchester Linear Law (area fire):

The basic hypothesis of the Linear Law is

$$\frac{dx}{dt} = -axy \qquad \text{and} \qquad \frac{dy}{dt} = -bxy$$

While a and b are still referred to attrition coefficients, they differ form those coefficients used in the Square Law. Specifically, the attrition coefficients are measured in units of (casualties/ ((time) × (firers) × (targets)). Any comparison of attrition coefficients between laws is a comparison of applies with oranges, since the units are different. While the Linear Law is usually assumed to apply to area fire weapons such as artillery, any other assumptions as well as the number of firers, would do as well.

Derivation of the State Equation:

The instantaneous casualty exchange ratio for the Linear Law can be expressed as:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-bxy}{-axy} = \frac{b}{a} \qquad (1.4.1)$$

In other words, the rate of change of y with respect to x is a constant, as in ancient warfare. Therefore x and y must always be related by

$$b[x_0 - x] = a[y_0 - y], \qquad (1.4.2)$$

Where x_0 and y_0 are the initial values of x and y. If the battle breakpoints are x_{BP} and y_{BP}, the fighting strengths of the two sides are now $b(x_0-x_{BP})$ and $a(y_0-y_{BP})$, respectively. The Linear Law derives its name from these formulas, since fighting strength is linear in the number of combatants. The winner is still the side with the larger fighting strength. Note that 1.4.2 is the same state equation that holds in ancient warfare; the dynamics change drastically under the Linear Law, but not the final outcome.

1.5 Other Functional Forms of Lanchester`s Original Models:

We have shown how Lanchester`s differential equations can be used to answer some basic questions on combat under the conditions of aimed and area fire homogeneous combat.

However, as previously noted, combat is rarely homogeneous and as the original Lanchester models while useful have many shortcomings. Some of these shortcomings are:

- considers only constant attrition rate coefficients.
- no force movement during battle.
- battle termination is not modeled.
- tactical decision processes are not considered.
- C^3 is not considered.
- no logistical aspects are portrayed.
- suppressive effects of weapons are not considered.
- target prioritization/ fire allocation not explicitly considered.
- Non combat losses are not considered.

Mixed Combat

Up to this point we have developed the necessary functional forms for the aimed fire(or Square Law) and the area fire/ near Law equations for homogeneous force combat. Form here it is possible to describe various forms of combat that are combinations of these two forms. The most obvious form is the one for **mixed** combat. Specifically where one force uses aimed fires and the opposing force uses area fires, denoted F\FT.

This situation is analogous to the X force attacking the Y force in a prepared defensive position. While both sides use aimed fires, it is important to remember that the time to acquire a target for an X firer dominates the attacking force actions and therefore the Linear Law applies. The same situation was shown to apply for insurgency operation models where one force ambushes another force. If X is in the open and Y ambushes X, then the Y firers use aimed fires but the X force fires must use area fire since they do not know the exact positions of their attackers. In these cases, the state equation can be developed exactly as in the linear and Square Law to yield:

$$\frac{b}{2}(x_0^2 - x^2) = a(y_0 - y) \qquad (1.5.1)$$

Logarithmic Law (non combat losses):

A second extension of the Lanchester models hypothesizes that the initial states of a small unit engagement can be models as a T/T attrition process or

$$\frac{dx}{dt} = -ax \quad \text{and} \quad \frac{dy}{dt} = -by \qquad (1.5.2)$$

This process is referred to as the logarithmic law from its state equation

$$b\ln\frac{x_0}{x} = a\ln\frac{y_0}{y} \qquad (1.5.3)$$

The logarithmic law is almost silly as a ((combat)) model because each side decreases asymptotically to zero independent of number of combatants on the other side. ((We have met the enemy, and he is us!)) But the logarithmic law makes more sense than might appear at first sight, since there are many sources of attrition other than hostile fire that must be accounted for (disease, desertion,…). The logarithmic law is not the whole story, but including terms such as-ax in the expression for dx/dt can still be used to model such phenomena in a larger situation.

Helmbold Equations:

A general form for homogeneous force attrition rate that yields the square, linear, and logarithmic laws as special cases was postulated by R. Helmbold in 1965. He stated that the relative fire effectiveness is influenced by the force ratio in the sense that if x/y is extremely large. Then X cannot effectively bring all his weapons to bear on the Y force. His reasoning was based on the perception that limitations of space, terrain masking, and the target engagement opportunities would prevent a large force from using its full firepower.

In conjunction with this hypothesis, Helmbold suggested that the following Lanchester- type differential equations would be more appropriate

$$\frac{dx}{dt} = -a\left(\frac{x}{y}\right)^{1-w} y \quad \text{and} \quad \frac{dy}{dt} = -b\left(\frac{y}{x}\right)^{1-w} x \qquad (1.5.4)$$

Where ω is a measure of efficiency with which the large force can be brought to bear on the small force. The alert reader will immediately see that these equations are the aimed fire equations with a force ratio modifier added in.

In order to illustrate the range of situations that the Helmbold equations cover, we need only to assign ? the values of 0,1,and ?and evaluate the resulting differential equations.

When

- ω=0, the Logarithmic Law.
- ω=1, the Square Law.
- ω=1/2, the Linear Law, at least for the state equation.

In the last case, Helmbold`s equations share with ancient warfare the property that a force can be annihilated in finite time. This is not true for the Linear Law, but nonetheless the same state equations holds.

1.6 Enrichment to Lanchester- type Models:

The preceding section dealt with modeling the attrition processes for various combat situations in terms of force characteristics for homogenous force combat. While these equations can cover a broad range of combat scenarios, they do not account for many factors that can affect the outcome of a battle. Since we recognize that the dynamics of battle entails a myriad of factors, we must introduce them into our earlier equations if we wish to accurately portray combat. In this section we will consider the following enrichments of the Lanchester – type models:

1. replacements, reinforcements, and\or withdrawal,
2. range dependency,
3. heterogeneous forces, and

Replacements, Reinforcements, and Withdrawal:

Each of these there options is a reality on the battlefield and in practice, a critical decision problem faced by a commander. While each alternative may occur under various conditions and in different form, we will only consider the simple and direct changes. For our purpose, we define two models:

- continuous replacement/ withdrawal, and
- unit reinforcement.

The continuous model simply adds a constant to each equation to represent replacement or withdrawal at a specified rate. Morse and Kimball (1950) define P and Q to be the Logarithmic Law attrition:

$dx/dt = P - ay - \beta x$ and $dy/dy = Q - bx - \alpha y$ (1.6.1)

They are able to obtain a complicated analytic solution. We will not discuss it further, except to note that x(t) or y(t) can now be an increasing function of time on account of the reinforcements. Indeed, the equations were motivated by prior work in biological systems where increases are natural.

In the ease of unit reinforcement, instantaneous changes in x(t) or y(t) occur at a particular time (t_r) which is the reinforcement/ withdrawal time. Unlike the continuous model, this process occurs outside the basic attrition equation. This essentially requires us to stop the equation at t_r and resume the battle with a different force structure. Figure 1.6.1 illustrates the unit replacement process over time.

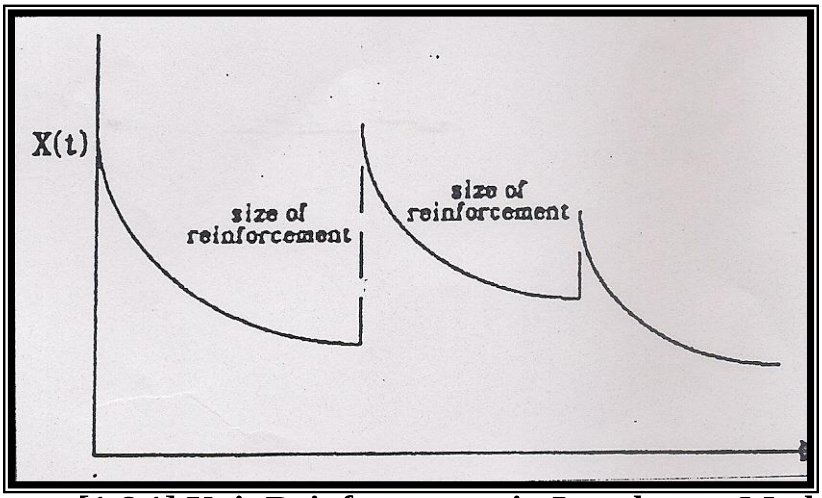


Figure [1.6.1] Unit Reinforcement in Lanchester Models

Range Dependency

Early models failed to consider in detail the effect of range on the attrition process. Practical experience indicated attrition is affected by range and should be considered depending on the resolution level of the model. Under range dependency, the attrition coefficients are functions of range and the differential equations are of the form

$$\frac{dx}{dt} = -a(r)y \quad \text{and} \quad \frac{dy}{dt} = -b(r)x$$

The dependency of the attrition coefficient on range was first studied by Bonder for a constant speed attack and various forms for a(r) and b(r). Based on his studies, Bonder suggested that constant attrition coefficients could be replaced by

$$a(r) = a_0\left(1 - \frac{r}{r_{\max}}\right)^{\mu} \quad for \quad 0 \le r \le r_{\max}$$

$$a(r) = 0 \quad for \quad r \ge r_{\max} \tag{1.6.2}$$

Where r_{max}=maximum range of the weapon system, and a_o=maximum attrition rate.

Plotting the attrition coefficient as a function of range (Figure 1.6.2) we see how different values for μ can affect the outcomes.

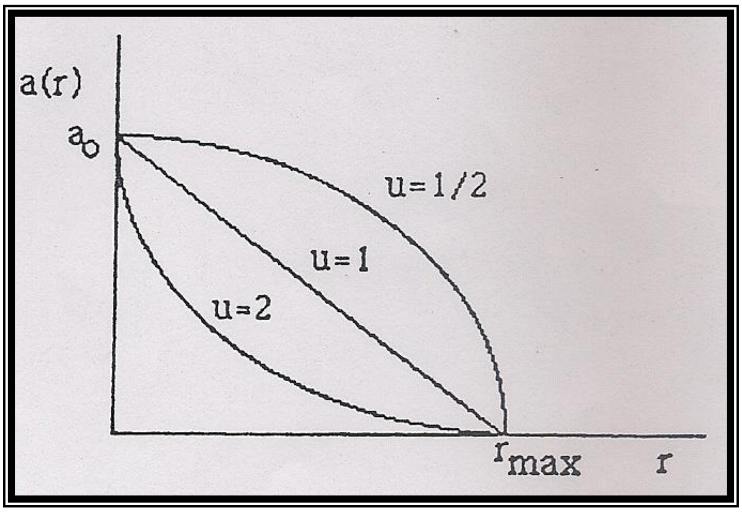


Figure [1.6.2] Bonder Range Dependent Attrition Coefficient plots

When we consider the constant speed model, we can express the range as a function of time.

$$r(t) = r_0 - vt$$

Where r_0 = initial range and v = the closing velocity.

The differential equations then become functions of t only

dx/dt = - a[r(t)]y

And can be analyzed.

Heterogeneous Forces:

Up to now we have assumed that individual elements of the X force have identical characteristics. Thus only the total number of combatants X(t) is the driving factor for attrition assessment. A schematic of homogeneous combat is shown in Figure 1.6.3.

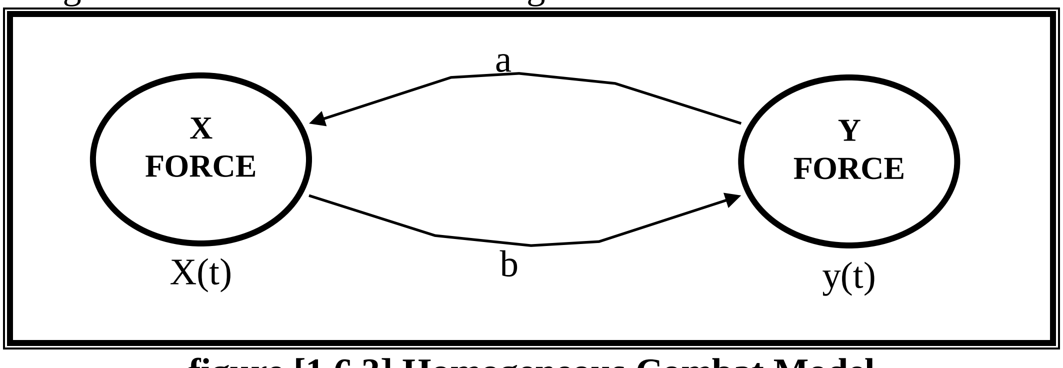


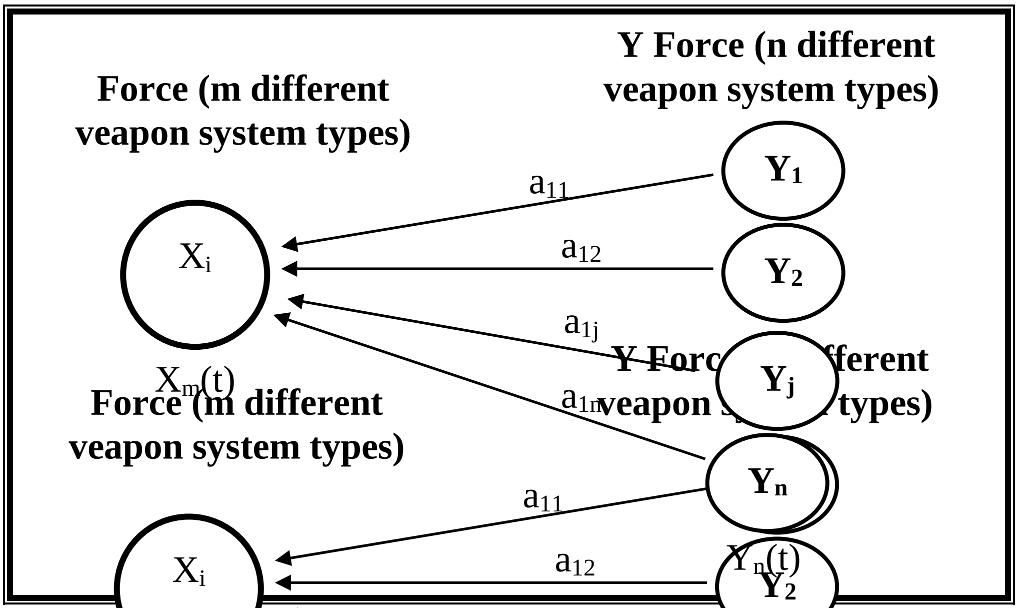
figure [1.6.3] Homogeneous Combat Model

Now let us consider a combined arms force:

$X = [x_1(t), x_2(t), \ldots x_m(t)]$

Y =[y_1(t), y_2(t),…..y_n(t)]

Where x_i(t) = number of X survivors of weapon system i at time t



Figu[illegible] Heterogeneous Combat x_i system vs. Y force

Lanchester-type model for attrition assessm[illegible]ll involve m+n differential equations- assessing the casualty rate for [illegible] x_i,y_i separately. If we select a single x_i system and consider att[illegible]o that system it would resemble Figure 1.6.4

Therefore, we can assess the attrition to a single system as:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \quad \text{(attrition of } x_i \text{ systems caused by } y_i \text{ system)}$$

(Note: the attrition may be 0 for some j if y_i does not kill x_i.)

For the heterogeneous model to function we have to make two assumptions about additivity. The first assumption, additivity says that there is no direct synergism. Simply stated the only way any antitank systems can contribute to the effectiveness of tank systems is by killing enemy tank systems. Consequently, their presence or absence in a force does not enhance the killing potential of a tank system. Hence synergism does not exist if attrition depends only on y_i. If attrition depends on y_i and y_k, for k≠j then synergism exists. To model synergistic effects is a complex task however it is not a problem here as the additivity assumption has eliminated the possibility of such effects.

The second assumption, proportionality says that the loss rate of x_i caused by y_i is proportional to the number of y_i that engage x_i. To better understand this assumption let us define ψ_{ij} as the fraction of y_i fires allocated to targets of type x_i where $\left(\sum_i \psi_{ij} = 1 \right)$. Then on the average we can say that:

$$y_{ij} = \psi_{ij} y_j$$

Is the number of y_j`s that engage x_i. For example, if y_i=100 and ψ_{ij} = 0.25 then:

y_j = 25

This does not say that only 25 Y firers shoot at x_i, but rather averaged over the y_i force, 1/4 of the time is spent engaging x_i targets.

Now if we let a_{ij} represent the attrition rate of one y_j system shooting at x_i, then if all the y_i firers are allocated against x_i systems

$$a_{ij} \psi_{ij} y_j = a_{ij} y_{ij}$$

Defining the combination of the attrition term (a_{ij}) and the allocation term ψ_{ij}, we get:

$$A_{ij} y_j = a_{ij} y_{ij}$$

Since we now can represent one system within the force, it is a simple step to model the complete system. Hence the complete heterogeneous system is

$$\frac{dx_i}{dt} = -\sum_{j=1}^{n} A_{ij} y_j \quad \text{i= 1,…,m}$$

$$\frac{dy_i}{dt} = -\sum_{j=1}^{m} B_{ji} x_i \quad \text{j= 1, …,n}$$

With initial conditions $x_i(0) = x_i^0; y_j(0) = y_j^0$ And with the understanding that A_{ij} becomes zero if either x_i(t) =0 or y_i(t)=0.

Once we have very written this system of equations, we can go no further analytically. We are at a point very similar to the original Square Law solution but the answer gives little insight into the combat dynamies. In short, the equations are too complex and there are too many coefficients. This leaves us with the problem of application to real world models.

If we hold A_{ij} and B_{ij} constant, the equations are essentially the Lanchester Square Law equations. However, we are not bound by any one particular law when we model heterogeneous force combat. In most operational combat models using Lanchester –type attrition processes, the heterogeneous equations shown above are either explicitly or implicitly changed to correspond to the nature of particular system interactions. Thus in a series of i_j engagements there option occurs when the form of the basic equation is changed by letting the coefficients be variable and letting A_{ij}/B_{ij} be functions of the number of x_i`s and $y_i$`s. In either case, we are foreed to use numerical solutions or some method gor coefficient estimation.

While heterogeneous force combat appears to be a nearly impossible task to model, it is quickly placed in perspective if one remembers that same techniques used to model homogeneous combat can be used to model the subcomponents of heterogeneous force combat. Thus we may state, simply, that heterogeneous force combat is just the summation of a series of homogeneous force battles.

Stochastic Lanchester Models Probability Maps

Let P(m,n,t) be the probability that the state is (m,n) at time t. A probability map simply shows the probability for all states at some specified time. Probability maps can be constructed by taking advantage of the fact that P(m,n,t) must satisfy the Chapman – Kolmogorov equations (ref to Ross):

$$\frac{dP(m,n,t)}{dt} = A(m+1,n,t)P(m+1,n,t) + B(m,n+1,t)P(m,n+1,t)$$
$$-(A(m,n,t)+B(m,n,t))P(m,n,t)$$

Where A() and B() are the transition rates. If the battle starts in state (m_0,n_0) then m and n can be confined to $0 \le m \le m_0$ and $0 \le n \le n_0$, with P(m,n,t) being 0 otherwise. Since the state (0,0) is also impossible, there are a total of $m_0n_0 + m_0 + n_0$ simultaneous differential equations that must be solved. Figures 1.6.6-8 show some examples. The figures apply to a Square Law battle where A(m,n,t)=.01n, B(m,n,t)=.2m, m_0=20 and n_0=40. The deterministic version would have the y-side winning at time 62.32 with 28.28 survivors. The figures show probability maps at time 20,40 and 60 for the stochastic version. The battle may actually be over by time 40, since at that time some probability has already accumulated in the states where m=0. By time 60, most of the probability is in those. In a less lopsided battle, there might also be some probability in the states where n=0, but in this case the m-side has essentially no chance of winning. Note that the deterministic version gets the winner right, and (by eyeball) the average number of survivors, but Figure 1.6.8 makes it clear that the number of n- survivors can vary quite a from its average.

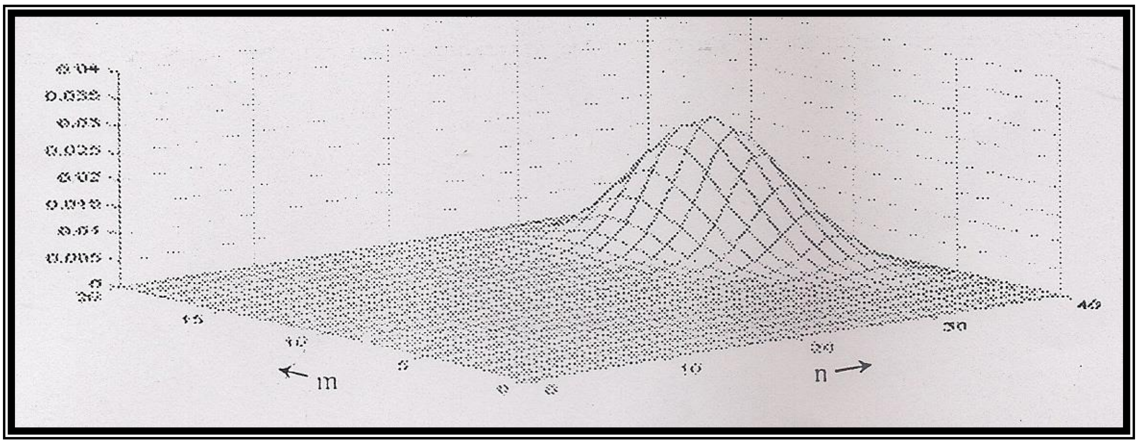
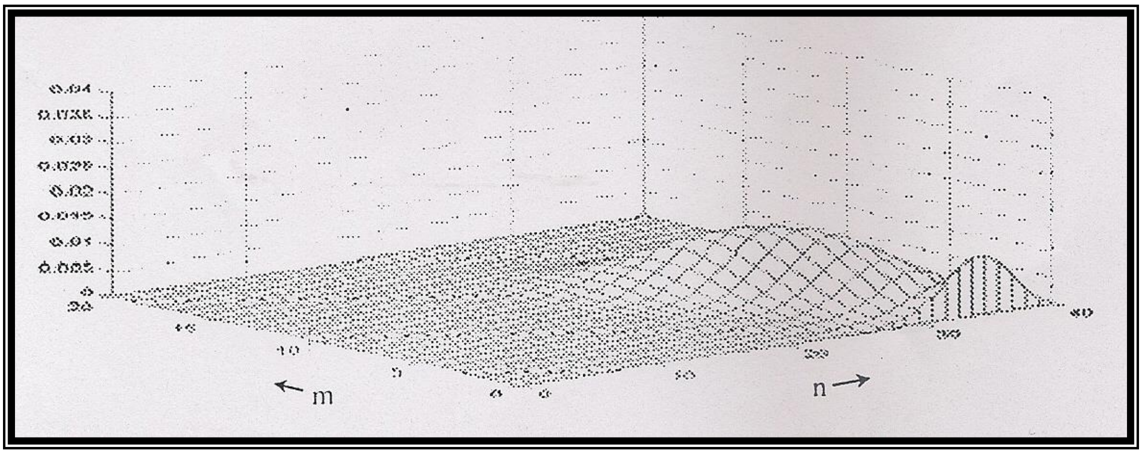


Figure [1.6.6] Probability Map at Time 20



Figure[1.6.7] Probability Map at Time 40

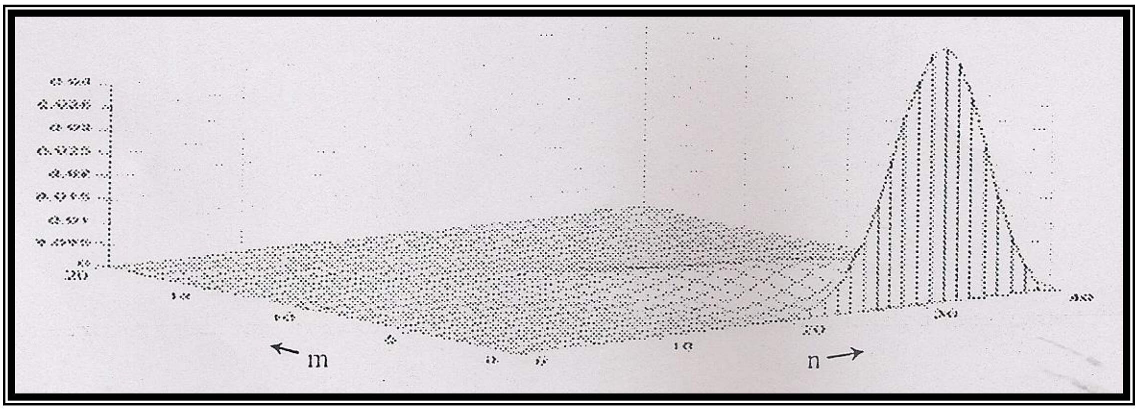


Figure [1.6.8[Probability Map at Time 60

1.7- Attrition Coefficient Estimation for Lanchester Models:

Throughout the chapter we have referred to casualty rates or attrition coefficients while providing only simple dimensional definitions. Additionally we have assigned values to the coefficients for purposes of examining trends and combat processes but never have we shown how we got the values for a and b. For illustrative purposes let us consider a simple F|F deterministic combat process. By definition we know that:

$$\frac{dx}{dt} = -ay \quad \text{and} \quad \frac{dy}{dt} = -bx$$

Where a=X casualties/ unit time/ Y firer and b=Y casualties/ unit time/ X firer. And we can say that the attrition coefficients a and b are a function of some unspecified attrition factors (a= f(attrition factors)).

Inherent in these hypotheses is the total X casualties per unit of time is proportional to the number of Y firers. Intuitively we know that many other factors influence attrition. This raises the question of how to capture these other factors into the attrition rate coefficients, a and b. If we are modeling a battle in which any these factors change with t (e.g. range) then we must let a= a(t), a nonconstant. Which our first reaction to this is to say that this will lead to increasingly complex and intractable equations, recall that by using numerical solution techniques our task will not become any harder since Δt Should be sufficiently small for a(t) to be considered constant within the interval. Therefore incorporating time dependent factors into attrition coefficients need not be avoided for fear of complexity.

As indicated above, the prime consideration for the modeler is defining the time unit to be used. For example. If we let one time step equal one day (as is the usual practice in highly aggregated firepower score models) then a is measured in casualties/ day/ enemy. But combat is not a uniform process over an entire day. Thus we somehow have to average attrition over various battle phases including parts of the day when non-direct combat engagements are occurring. On the other hand if we let of the time step equal one minute then we need 24×60 or 1440 time steps to make-up a sing day. In each time step we can compute a, b to reflect the essentially instantaneous combat conditions. Concurrently, the model simulation then sets up the conditions or situation between opposing forces and from the situation we can compute new values for a and b for use in the next time step. Therefore we can relate a,b directly to weapon systems parameters such as P_k, firing rate, basic loads, etc, through a series of look- up tables.

Operational models currently in use tend toward the second ease, using time steps for ground casualty assessment in the range of 0.1 to15 minutes. For our purposes, coefficient estimates will concentrate on small? So we can assume

1. a,b are essentially constant over the interval (t+Δt).
2. Δt is small enough that the battlefield conditions (including force size) at the beginning of the interval are representative of the entire interval.

As with any attempt to model real world phenomenon it is logical to start with a simple representation and then enrich or embellish as necessary. Using this approach will allow us to build the necessary foundation for more sophisticated techniques without losing sight of our purpose of how to estimate attrition coefficients. With the direction for the examination of coefficient estimation set, let us first look at the basic technique using a deterministic model.

Naive Estimate

In the naïve estimate we consider the casualty rate, a, for point to be:
a = (firing rate) × (prob. Of a casualty per shot)
or
$a = v_f \times P_{ssk}$

Where the maximum value for v_f is based on engineering parameters while the average v_f is almost always less due to battlefield conditions developed essentially from behavioral data.

The P_{ssk} is a single shot kill probability based primarily on engineering data and dependent upon factors such as range, target type, and firer posture. For aimed fire we consider P_{ssk} to be constant and firing dominates the target acquisition process then V_f is also constant. Thus in the aimed fire case, we get

$\frac{dx}{dt} = -ay$ where $a = v_f P_{ssk}$ is a constant.

In the case of area fire, V_f being constant is a reasonable assumption. The probability of a single shot kill is usually determined by comparing the lethal area of a round to that of the target area. Then
P_{ssk} = expected number of targets killed by one round.

Subsequently the probability of a kill for single shot can be expressed as:

P_{ssk} = lethal area of one round time the target density

$$P_{ssk} = a_1 \frac{x}{A_{tgt}}$$

$$= \frac{a_1}{A_{tgt}} x$$

Where a_l= lethal area of one round
A_{tgt}= total target area
And x = number of targets.
Thus

$$\frac{dx}{dt} = v_f \frac{a_1}{A_{tgt}} xy$$

Yielding the expression for the Linear Law since a depends up on x.

Poisson (Markov) Assumption

Recalling from the earlier discussion of stochastic attrition models that we were able to determine the outcome of battle based on the time between casualties and several factors. During this investigation we noted that the casualty rate could be expressed as the reciprocal of the expected time between casualties at any time during the battle. Therefore since we can express the attrition coefficient as

$$a = \frac{1}{E[T_{xY}]}$$

Where the denominator is the expected time for one Y firer to kill one X target, we can estimate the attrition rate coefficients throughout a battle if we can develop a model for the expected time to kill a target.

Such an approach is preferred because we can easily incorporate the various factors that are relevant to the weapon firing cycle that were merely averaged together in the naïve estimate. Analogously, for heterogeneous Lanchester models we can compute attrition rates as:

$$a_{ij} = \frac{1}{E[T_{ij}]}$$

Where T_{ij} is the time (a random variable) for one Y_j firer to kill one passive X_i target in an engagement where Y_j concentrates on X_i. The level of concentration can then be modified by a fire allocation factor (ψ_{ij}) based on acquisition priorities. This specific process will be examined in greater detail when we discuss the Bonder methodology for attrition coefficient generation.

Chapter two
Stochastic Model Approach

2.1 The stochastic combat model

The state of each air combat is the important parameter that must be represented at any instant of time. A combat state means the number of the remaining aircrafts of the two forces at that time. If the two opposing forces are named as X and Y and each of which beginning a combat with his initial force size (i.e.x_0 and y_0 of aircrafts for each side respectively). Each one of the two forces will terminate the battle if his remaining number of aircrafts reaches an indicated number called break point (or terminated) number. These two number are assumed to be x_f for the X force and y_f for Y force. The possible states of these two forces can be represented as shown in figure (2-1).

The states of the two force is (x_0,y_0) at the beginning time of the combat. As the combat time increase the chance of decreasing the two forces will increase. This is due to the casualties (attrition or kill rate) of each one on the other. This reduction happens in a stochastic manner. The side reaches the termination number firstly will lose the combat. the remaining number of aircrafts on the two sides at that time will represent the expected survival number. So the probability of being in each state at each time (t), must be evaluated. This probability is denoted by $P_{x,y}(t)$

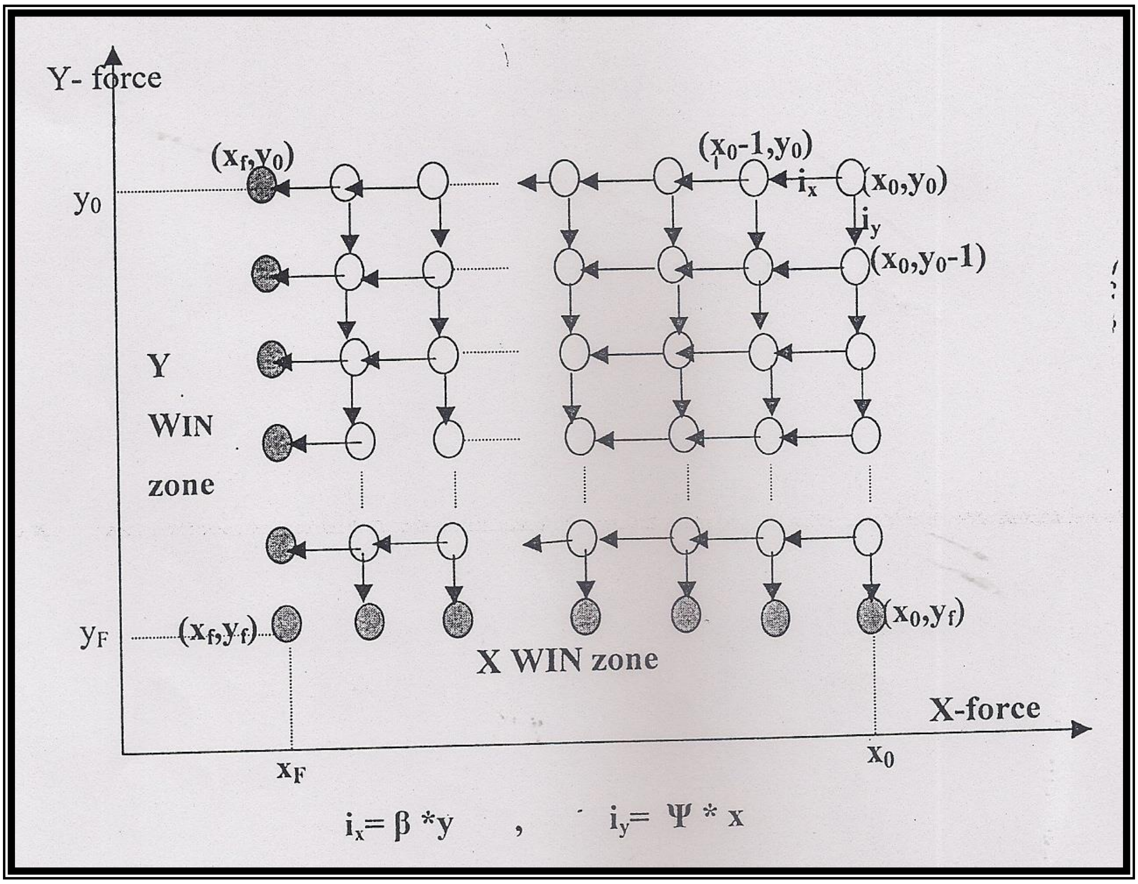


Fig [2.1] state space lattice representation of a Homogeneous Stochastic air combat

Conuting process is either the killing process associated with one side or the killing process associated with the other side. Any stochastic process like {X(t), t≥0} Is a counting process if X(t) represents the number of events that have occurred up to time t {assuming X(0)=0] and:

$$\Pr\{X(t+\Delta t)-X(t)>1\}=O(\Delta t) \qquad (2.1.1)$$

i.e. the probability of one side having two or more kills during a small time interval (t, t+ Δt) is proportional to O (Δt)

2.2 Renewal Process

Renewal process was defined as a discrete- state process, which counts the number of event occurring, it may be called a counting process. If the time intervals between consecutive epochs of occurrence of the events be independent and identically distributed random variables. Then N(t) is called a renewal counting process.

The time parameter can be considered to be either discrete or continuous. For a given value of t, N(t) is a proper random variable, and its distribution is:

$$P_n(t) = P[N(t) = n] = F_n(t) - F_{n+1}(t) \quad n = 0,1,2.... \qquad (2.2.1)$$

And

$$E(N(t)) = U(t) = \sum_{n=1}^{\infty} F_n(t) \qquad (2.2.2)$$

Where:

$$F_n(t) = F_1(t) * F_{n-1}(t)$$

The renewal function and the renewal density integral equations are:

$$U(t) = F_1(t) + \int_0^t U(t-\tau)dF(\tau) \qquad (2.2.3)$$

$$u(t) = f_1(t) + \int_0^t u(t-\tau) f(\tau)d\tau \qquad (2.2.4)$$

2.3 Recurrent and Transience

Let fi = the probability of returning to state I given that X_0=i

$$f_i = P(X_n = i \quad for\ some\ n \geq 0 / X_0 = i)$$

i is recurrent if f_i =1 and that i is transient if f_i <1

If i is transient, the number of time the process visit I is a geometric random variable with probability of success equal to fi. i is transient if the expected number of visits to i is finite, and if i is recurrent then expected number of visits to I is infinite.

i is transient if $\sum_K P_{i,i}^k < \infty$

2.4 Markov- Chains

Let X =(X_0,X_1,…) is a sequence of random variable taking values in a countable set S called the state space of X. saying that X is markov chain if one have[13]:

$$P(X_{n+1} = j / X_n = i, X_{n-1} = i_{n-1},....., X_0 = i_0) = p_{ij}$$

Setting n=0:

$$P(X_1 = j / X_0 = i) = P_{ij}$$

However

$$P(X_2 = j / X_1 = i) = \frac{P(X_2 = j / X_1 = i)}{P(X_1 = i)}$$

$$= \sum_k \frac{P(X_2 = j, X_1 = i, X_0 = k)}{P(X_1 = i)}$$

$$= \sum_k P(X_2 = j, X_1 = i, X_0 = k) P(X_0 = k / X_1 = i)$$

$$= \sum_k P_{ij} P(X_0 = k / X_1 = i)$$

$= P_{i,j}$

In the same fashion, for any non – negative integer n,

$$P(X_{n+1} = j / X_n = i) = P_{ij} \qquad (2.4.1)$$

For this reason, the numbers P_{ij} are called transition probabilities, and

$$\sum_j P_{ij} = 1 \qquad (2.4.2)$$

2.5 Attrition rate

This factor is depending on the firing rate of each side during each time and measured by the counter effects. It means that one side is making a kill during the time interval (t,t+Δt) given that the combat is in state (x, y) at time t [17, 3].

The term Pr(S(t, x, y)) means: the probability that the combat is in state (x, y) at time t. i.e Pr(S(t, x, y)) = $P_{x.y}(t)$

The kill rate of aircraft x_i at time t is defined [3] as:

$$W_{x1}(t) = \lim_{\Delta t \to 0} \frac{\Pr\{Kx_1\}}{\Delta t} \qquad (2.5.1)$$

Where k_x means that x makes a kill during (t, t+ Δt)

If Δt is small enough, then:

$$\Pr\{K_{x1}\} = W_{x1}(t)\Delta t + 0(\Delta t)$$

$$W_{x1}(t/V) = \lim_{\Delta t \to 0} \frac{\Pr\{Kx_1 / V\}}{\Delta t} \qquad (2.5.2)$$

$$\Pr\{K_{x1}[K_{x1}(t,n)\} = i_x(t/n)\Delta t + 0(\Delta t) \qquad (2.5.3)$$

$$\Pr\{K_{x1} / S(t,x,y)\} = i_x(t/x,y)\Delta t + 0(\Delta t) \qquad (2.5.4)$$

Where

$i_d(t/n)$: be the kill rate of the side d at time t, given that it makes n kills up to time t, d= x, y.

The expression for $i_x(t/n)$ was derived by Jin [15] as:

$$i_x(t/n) = \frac{f_{n+1}(t)}{F_n(t) - F_{n+1}(t)} \qquad (2.5.5)$$

And

$$f_n(t) = \int_0^t f_{n-1}(t) f(t-u)du \quad for \;\; n \geq 2 \qquad (2.5.6)$$

$$F_n(t) = \int_0^t F_{n-1}(t-u)dF(u) \quad for \;\; n \geq 2 \qquad (2.5.7)$$

Where $f_n(t)$ is the probability density function of the interkilling time, and $F_n(t)$ is the corresponding distribution function.

To develop a certain expression for the kill rate in eq. using the appropriate distribution for such process. Most of the available literatures indicate that the exponential distribution is the appropriate. We need to derive an expression to represent.

3.6.1 The Kolmogorov Differential equations

As in discrete time:

$$P_{ij}(S+t) = \sum_k P_{ik}(s)P_{kj}(t)$$

And in matrix terms

P(S+t) = P(s) P(t)

Setting t = ds gives

P(s+ds) = P(s) P(ds)

Or

P(s+ds) – P(s) = P(s) [P(ds) – I]

Which gives

$$P'(s) = P(s)Q$$

Where

Q= P(0), and P(0) = I

This is called the kolmogorov forward equation.

$$P(X(S+ds)=j) = \sum_k P(X(S+ds)=j/D(s)=k)P(X(s)=k)$$

$$= \sum_{k\neq j} P(X(S)=k)q_{kj}ds + (1-\sum_{k\neq j} q_{kj})P(X(s)=j)$$

If:

$$q_{kk} = \sum_j q_{kj} \text{ then}$$

$$\frac{d}{ds}P(X(s)=k) = \sum_k P(X(s)=k)q_{kj}$$

Putting S = dt in the Chapman- Kolmogorov equations:

$$P'(s) = P(s)Q$$

It is theoretically possible to solve Kolmogorov equations, a solution is

$$P(t) = \exp(Qt) = \sum_n t^n Q^n / n!$$

Using fig (3-1) and the above definitions, the exact kolmogorov equations for the combat states can be derived [51] as follows:

a. For the initial state (x_0, y_0):

$$P_{x0,y0}(t+\Delta t) = P_{x0,y0}(t)[1 - x_0 i_x(t/x_0,y_0)\Delta t][1 - y_0 i_y(t/x_0,y_0)\Delta t]$$

$$= P_{x0,y0}(t)[1 - x_0 i_x(t/x_0,y_0)\Delta t - y_0 i_y(t/x_0,y_0)\Delta t]$$

$$= P_{x0,y0}(t) - P_{x0,y0}(t)\Delta t[x_0 i_x(t/x_0,y_0) + y_0 i_y(t/x_0,y_0)]$$

$$P_{x0,y0}(t+\Delta t) - P_{x0,y0}(t) = -P_{x0,y0}(t)\Delta t[x_0.i_x(t/x_0,y_0) + y_0.i_y(t/x_0,y_0)]$$

$$\lim_{\Delta t - 0} \frac{P_{x0,y0}(t+\Delta t) - P_{x0,y0}(t)}{\Delta t} = -P_{x0,y0}(t)[x_0.i_x(t/x_0,y_0) + y_0.i_y(t/x_0,y_0)]$$

$$\frac{dP_{x0,y0}(t)}{dt} = -P_{x0,y0}(t)[x_0.i_x(t/x_0,y_0) + y_0.i_y(t/x_0,y_0)] \qquad (2.62)$$

b. For the second state (x_0,y) where $y_f < y < y_0$, which is the nearest to the initial state (X side kills one from Y side first):

$$P_{x0,y}(t+\Delta t) = P_{x0,y}(t)[1 - x_0.i_x(t/x_0,y)\Delta t]$$

$$[1 - y.i_y(t/x_0,y)\Delta t] + P_{x0,y+1}(t)[x_0.i_x(t/x_0,y+1)\Delta t)]$$

$$\lim_{\Delta t - 0} \frac{P_{x0,y}(t+\Delta t) - P_{x0,y}(t)}{\Delta t} = P_{x0,y}(t)[x_0.i_x(t/x_0,y) + y.i_y(t/x_0,y)]$$

$$+ P_{x0,y+1}(t)[x_0.i_x(t/x_0,y+1)]$$

$$\frac{dP_{x0,y}(t)}{dt} = -P_{x0,y}(t)[x_0.i_x(t/x_0,y) + y.i_y(t/x_0,y)] + P_{x0,y+1}(t)[x_0.i_x(t/x_0,y+1)] \qquad (2.64)$$

c. For states (x, y_0) where $x_f < x < x_0$. The other side nearest state or (Y side kills one from the X side first):

$$\frac{dp_{x0,y}(t)}{dt} = -P_{x,y0}(t)[x.i_x(t/x,y_0) + y_0.i_y(t/x,y_0)] + P_{x+1y0}(t)[y_0.i_y(y/x+1,y_0)] \qquad (2.65)$$

d. For states (x,y) where $x_f < x < x_0$ and $y_f < y < y_0$, (this state is called transient state).

$$\frac{dp_{x,y}(t)}{dt} = -P_{x,y}(t)[x.i_x(t/x,y) + y.i_y(t/x,y)] + P_{x,y+1}(t)[x.i_x(t/x,y+1)] + P_{x+1,y}(t)[y.i_y(t/x+1,y)] \qquad (2.66)$$

5. For states (x, y_f), where $x_f < x < x_0$

$$\frac{dp(t)}{dt} = +P_{x,yf+1}(t)[x,i_x(t/x,y_f+1)] \qquad (2.67)$$

6. For states (x_f, y) where $y_f < y < y_0$

$$\frac{dP_{x,y}(t)}{dt} = +P_{xf+1,y}(t)[yi_y(t/x_f+1,y)] \qquad (2.68)$$

The initial conditions are

$$P_{x0,y0}(0) = 1 \qquad (2.69)$$

$P_{x,y}(0) = 1$ for all other states.

2.6.2 Kolmogorov equations solutions:

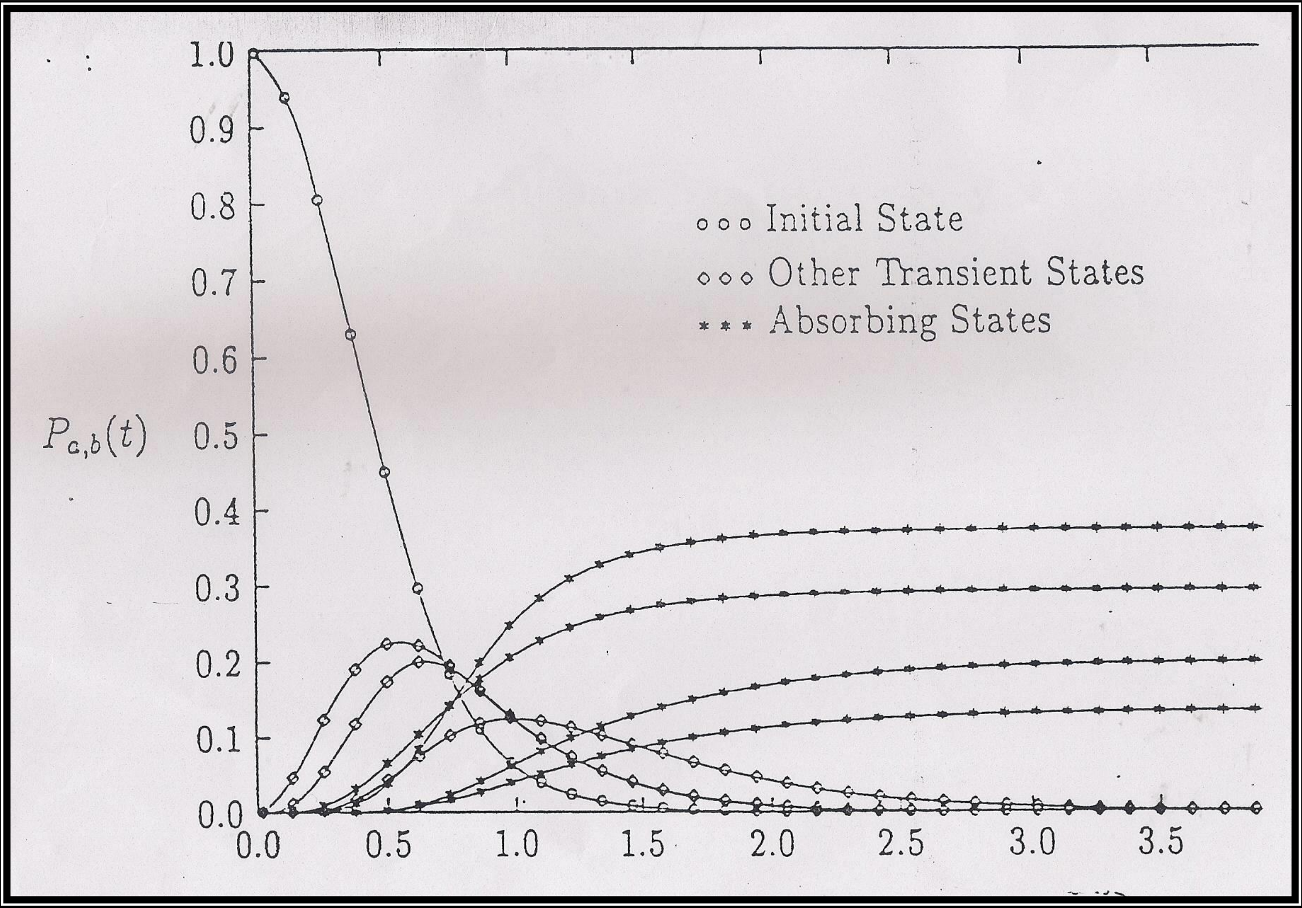
The set of differential equations (kolmogorov equations) can be solved if the conditional kill rate (attrition rate) is constant over the combat time. This is not true assumption and didn't give correct indications about the actual behavior of the participants. So the other way is to develop an approximation method for calculating such factor. This is a very difficult method and needs numerical calculations, which takes a very long time on the computer. This approach was done by[3]. He performs a numerical calculation using Range- Kutta- method to solve the set of Kolmogorov equations numerically by approximating the conditional kill rate in each state. He succeeded in approximating a conditional kill rate for combat states involving small numbers only (4 by 4). Any increase in the participant number will take very long time reach the answer. This approach wills strong the need for simulation as a tool to be used in such turbulent situations.

2.4 Discussion & Conclusion:

Everything we have done in the first part of this chapter has been deterministic and continuous, whereas discrete units fight actual battles very much subject to luck. Intuitively, the deterministic models should be at their best when the numbers of units involved are large.

Using Lanchester attrition models in the current warfare indicates the following shortcomings:

- Considers only constant attrition rate coefficients
- No force movement during battle
- Battle termination is not modeled
- Tactical decision processes are not considered
- Suppressive effects of weapons are not considered
- Non – combat losses are not considered.



Figure[2,2] Typical Solutions of State probabilities

In this chapter a review a bout the most applicable combat models was done. From this review a clear comparison between the use of deterministic and stochastic models was reached. Many researchers make a foundation that any deals with the new combat must be through the stochastic approach. This approach makes use of the available deterministic tools in evaluating some combat factors. Some of these factors are:

1. The weapon score and the force strength.
2. The termination rules and the force divisions.

3. The available number of sorties for each type of aircrafts at each unit of time.
4. the munitions needed for each target.
5. The optimum number of weapons (aircrafts) required to kill each target.

المستخلص

تشير نماذج الاستنزاف التفاضلية نوع **Lanchester** الى مجموعة من المعادلات التفاضلية التي تصف تغيرات مستويات قوة المعركة ومتغيرات اخرى مهمة في المعركة عبر الزمن.
تعبر النماذج المذكورة اعلاه عن الاصابات (**الخسائر**) من حيث حجم القوة ومتغيرات اخرى مرتبطة بها ، لقد كسب نموذج الاستنزاف نوع **Lanchester** اهمية من خلال اعطاءها معرفة وبصيرة للمعارك الديناميكية .
تناول البحث اشتقاق معادلة قانون التربيع الرياضي النار المصوبة **(Aim Fire)** لنموذج **Lanchester** والمعادلة الخطية **(Area Fire)** للنموذج ذاته وتمت حساب نتائج المعركة وحالة الانتصار لكل طرف من اطراف التنافس .
كما تناول البحث نماذج**Lanchester** للخرائط الاحتمالية العشوائية وقد تم حساب معادلات ومعدلات الاستنزاف لكل حالة من حالات الانتصار والهزيمة لكل متنافس وقد جرى الحل من خلال معادلات Kolmogrov كذلك معادلات Markov الانتقالية واستعرضت الاستنتاجات والنتائج .

Reference

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[2] David L. Bitters, "**Efficient Concentration of Forces, The Military Operations Research Society**", 1995.
[3] Yang. Jand Gafarian, "**Homogeneous Stochastic Combat, The Military Operations Research Society**", 1995.

```
CLS
DIM A(20, 20, 35)
DIM Y (50), W(50)

A0 = 4: B0 = 8: AF = 0: BF = 0: RA = .02: RB = .01
   TT = 34
   REM 1     P(A0,B0)
A1 = (A0 * RA + B0 * RB)
FOR K = 0 TO TT
       A(B0, A0, K) = EXP (- A0 * K)
NEXT K
REM 2 P(A0, B)
     FOR I = B0 – 1 TO BF + 1 STEP – 1
     A (I, A0, 0) = 0
     FOR K = 1 TO TT
          A1 = (A0 * RA + I * RB)
          FOR X = 0 TO K
             Y (X) = A(I + 1, A, X) * EXP (A1 * X)
      NEXT
                  REM CALCULATE THE WEIGHT
      IF K >1 THEN
      FOR II = 1 TO K – 1 STEP 2
         W (II) = 4
         W (II +1) = 2
      NEXT II
      END IF
      W(0) = 1: W(K) = 1
      SUM = 0
      FOR II = 0 TO K
        SUM = SUM + Y(II) * W(II)
      NEXT II
   SUM = SUM /3
   A(I, A0, K) = EXP (- A1 * K) * A0 * RA * SUM
NEXT K
NEXT I
            REM 3 P(A, B0)
FRO J = A0 – 1 TO AF + 1 STEP -1
            A (B0, J, 0) =0
  FOR K = 1 TP TT
   A1 = (J * RA + B0 * RB)
      FOR X = 0 TO K
        Y(X) = A(B0, J + 1, X) * EXP (A1 * X)
NEXT X
```

REM CALCULATE THE WEIGHT

```
IF K > 1 THEN
FOR II = 1 TO K – 1 STEP 2
  W (II) = 4
   W (II + 1) = 2
NEXT II
END IF
W (0) = 1: W(K) = 1
SUM = 0
FOR II = 0 TO K
    SUM = SUM + Y (II) * W (II)
    NEX II
    SUM = SUM / 3
```

A (B0, J, K) = EXP (- A1 * K) * B0 * RB * USM

```
NEXT K
NEXT J
```

REM 4   P (A, B)

```
FOR J = A0 – 1 TO AF + 1 STEP – 1
   FOR I = B0 – 1 TO BF + 1 STEP – 1
           A (I, J, 0) = 0
FOR K = 1 TO TT
          A1 = (J * RA + I * RB)
          FRO X = 0 TO K
Y (X) = (A(I + 1, J, X) * J * RA + I * RB * A(I, J + 1, X))
        NEXT X
```

REM CSALCULATE THE WEIGHT

```
            IF K > 1 THEN
            FOR II = 1 TO K – 1 STEP 2
        W(II) = 4
        W(II + 1) = 2
NEXT II
END IF
    W(0) = 1: W(K) = 1
    SUM = 0
    FOR II = 0 TO K
      SUM = SUM + Y (II) * W (II)
     NEXT II
SUM = SUM /3
```

A(I, J, K) = EXP (-A1 * K) * SUM

```
    NEXT K
  NEXT I
NEXT J
```

REM 5   P(A, BF)

```
     FOR J = A0 TO AF + 1 STEP – 1
A(BF, J, 0) = 0
```

```
FOR K = 1 TO TT
    FOR X = 0 TO K
        Y (X) = A (BF + 1, J, X)
   NEXT X
   REM  CALCULATE THE WEIGHT
             W(0) = 1
IF K > 1 THEN
FOR II = 1 TO K – 1 STEP 2
   W (II) = 4
   W (II + 1) = 2
NEXT II
END IF
W (K) = 1
SUM = 0
FOR II = 0 TO K
   SUM = SUM + Y (II)  W(II)
NEXT II
     SUM = SUM / 3
     A (BF, J, K) = J * RA * SUM
NEXT K
NEXT J

        REM 6      P (AF, B)
FOR I = B0 TO BF + 1 STEP – 1
   A (AF, I, 0) = 0
   FOR K = 1 TO TT
   FOR X = 0 TO K
        Y (X) = A(I, AF + 1, X)
        NEXT X
        REM CALCULATE THE WEIGHT
                 W(0) = 1
                  IF K > 1 THEN
                  FOR II = 1 TO K – 1 STEP 2
                      W(II) = 4
                      W(II +1) = 2
                 NEXT II
                 END IF
              W (K) =1
              SUM = 0
              FOR II = 0 TO K
               SUM = SUM + Y(II) * W(II)
              NEXT II
         SUM = SUM /3
         A(I, AF, K) = I * RB * SUM
    NEXT K
    NEXT I
REM PRINT
                         FOR K = 0 TO TT
```

```
A1 = 0
FOR J = B0 TO BF STEP – 1
FOR I = A0 TO AF STEP -1
A1 = A1 + A(J, I, K)
NEXT I
NEXT J
PRINT A1: INPUT YY                NEXT K
```